XI Jornadas Hispano-Lusas
de Matemáticas, vol 1, 226-230
Universidad de Extremadura, 1986

ON STRONG DUALS OF UNIFORMLY $\lambda(P)$-NUCLEAR SPACES

J. M. García Lafuente
Departamento de Matemáticas. Universidad de Extremadura. 06071-BADAJOZ. España

1. Introduction

Smooth sequence spaces $\lambda(P)$ of infinite type or G_∞-spaces were introduced in [7] by Terzioglu as a generalization of power series spaces $\Lambda(\alpha)$ of infinite type. The corresponding study of $\lambda(P)$-nuclearity is carried out in [6] , extending the notion of $\Lambda(\alpha)$-nuclearity of Ramanujan ([5]) .

In the present paper we give a condition on P and P' for a Köthe space uniformly $\lambda(P)$-nuclear has a strong dual $\lambda(P')$-nuclear, where $\lambda(P)$ and $\lambda(P')$ are nuclear G_∞-spaces. From this result one can deduce a condition, already known by Ramanujan ([5]) , about $\Lambda(\alpha)$-nuclearity of strong duals.

2. Definitions

In the sequel $\lambda(P)$ will be a nuclear G_∞-space and we refer to [6] for the not explained properties of $\lambda(P)$-nuclear spaces. We shall only recall the following characterization of $\lambda(P)$-nuclearity of a

Köthe space $\lambda(Q)$, wich comes from a corrected version due to Köthe, of a result of Brudovskii about s-nuclearity of $\lambda(Q)$ ([1])

Theorem 1 (Köthe-Pietsch-Grothendieck Criterion) A Köthe space $\lambda(Q)$ is $\lambda(P)$-nuclear if and only if for each $a \in Q$ there exists $b \in Q$ with $b_n \geq a_n$ for all $n \in N$, and an injection $\sigma: \mathbb{N} \longrightarrow \mathbb{N}$ with $\sigma(N) = \{n \in N,\ a_n \neq 0\}$ such that the following condition is satisfied

$$(a_{\sigma(n)} / b_{\sigma(n)})_n \in \lambda(P)$$

In some applications (see [2]) and specially in the study of dual spaces of Köthe sequence spaces, it is of great interest the existence of a "universal" permutation σ valid for every element $a \in Q$ in the previous theorem. This leads Köthe [8] to define a sequence space $\lambda(Q)$ to be uniformly $\lambda(P)$-nuclear if there exists a bijection $\Pi: \mathbb{N} \longrightarrow \mathbb{N}$ such that for each $a \in Q$ there exists $b \in Q$ and $c \in \lambda(P)$ such that $a_{\Pi(n)} \leq b_{\Pi(n)} c_n$ for every $n \in N$. Obviously each uniformly $\lambda(P)$-nuclear Köthe space is $\lambda(P)$-nuclear, and the converse is true under certain assumptions on P (see for example [6]).

3. Strong Duals

It is known that if $\lambda(P)$ is a nuclear barreled G_∞-space, then the strong topological dual $\lambda(P)'_b$ is uniformly $\lambda(P)$-nuclear ([6] , corollary 4.1.). For general Köthe spaces $\lambda(Q)$ we have furthermore

Theorem 2 Let $\lambda(P)$ and $\lambda(P')$ be nuclear G_∞-spaces such that there exists a sequence $\sigma \in \lambda(P')$ with $\sigma_n \neq 0$ for all $n \in N$ and $\sigma^{-1} = (\sigma_n^{-1}) \in P$. Then the strong topological dual of each barreled Köthe space $\lambda(Q)$ uniformly $\lambda(P)$-nuclear is $\lambda(P')$-nuclear.

Proof. By barreledness the strong topological dual $\lambda(Q)'_b$ is isomorphic to the Köthe space $\lambda(L)$, where $L = \{\xi \in \lambda(Q)\ ,\ \xi \geq 0\}$ (see [3]). Let $\Pi : \mathbb{N} \longrightarrow \mathbb{N}$ be the universal permutation given by uniform $\lambda(P)$-nuclearity of $\lambda(Q)$ and for each $\xi \in L$ let τ be the sequence given by $\tau_n = \xi_n(\sigma_{\Pi^{-1}(n)})^{-1}$. For each $a \in Q$ there exists $b \in Q$ and $c \in \lambda(P)$ such that $a_{\Pi(n)} \leq b_{\Pi(n)} c_n$ for all $n \in N$. Consequently we have

$$\sum \tau_n a_n = \sum \xi_n (\sigma_{\Pi^{-1}(n)})^{-1} a_n = \sum \xi_{\Pi(n)} \sigma_n^{-1} a_{\Pi(n)} \leq \sum \xi_{\Pi(n)} \sigma_n^{-1} b_{\Pi(n)} c_n$$

But the sequences $(\sigma_n^{-1} c_n)$ and $(\xi_{\Pi(n)} b_{\Pi(n)})$ are both summable because $\sigma^{-1} \in P$ and $b \in Q$. Thus $\sum \tau_n a_n < +\infty$ and, since obviously $\tau_n \geq 0$ for all $n \in N$, we conclude that $\tau \in L$. On the other hand, let us choose a strictly increasing map $\beta : \mathbb{N} \longrightarrow \mathbb{N}$ such that $\xi_n \neq 0$ for all $n \in \beta(\mathbb{N})$. Then, the non-zero entries of the sequence (ξ_n/τ_n) can be rearranged by means of the injection $\beta \circ \Pi$ into the sequence $(\sigma_{\beta(n)})$ wich belongs to $\lambda(P')$ because for each $a \in P'$

$$\sum \sigma_{\beta(n)} a_n \leq \sum \sigma_{\beta(n)} a_{\beta(n)} \leq \sum \sigma_n a_n < +\infty$$

(for the first inequality remember that each sequence in P' is increasing and for the last one we apply the hypothesis). Finally, from Theorem 1 it follows that $\lambda(L)$, and hence the strong dual $\lambda(Q)'_b$, is $\lambda(P')$-nuclear.

<u>Corollary 1</u> (Ramanujan) If α and β are exponent sequences such that (α_n/β_n) converges to ∞, then the strong topological dual of each Köthe space uniformly $\lambda(\alpha)$-nuclear is $\lambda(\beta)$-nuclear.

Proof. If $P = \{(k^{\alpha_n})_n, k \in \mathbb{N}\}$ and $P' = \{(k^{\beta_n})_n, k \in \mathbb{N}\}$, then $\lambda(P) = \lambda(\alpha)$ and $\lambda(P') = \lambda(\beta)$ are the corresponding nuclear G_∞-spaces associated to the sequences α and β (in fact they are power series spaces of infinite type). By hypothesis $(2^{-\alpha_n/\beta_n})$ converges to 0 and thus $(2^{-\alpha_n}) \in \lambda(P')$ (see [5] Lemma 2.). Since $(2^{\alpha_n}) \in P$, we can apply the previous theorem to prove straightforward the corollary.

<u>Corollary 2</u> If $P = \{(e^{n^k})_n, k \in \mathbb{N}\}$, then the strong topological dual of each Köthe space uniformly $\lambda(P)$-nuclear is s-nuclear.

Proof. It is known that $s = \lambda(P')$ where $P' = \{((n+1)^k)_n, k \in \mathbb{N}\}$. The corollary easily follows from the Theorem 2 noting that $(e^n) \in P$ and $(e^{-n}) \in \lambda(P')$ because for all $k \in \mathbb{N}$, $\sum e^{-n}(n+1)^k < +\infty$.

Since $\lambda(P)$ in the previous corollary is not a power series space ([2] Theorem 2.25.), Theorem 2 supplies a stronger tool than [5] Proposition 8 to investigate s-nuclearity of strong duals.

REFERENCES

[1] Brudovskii, V. S. Associated nuclear topology, mappings of type s and strongly nuclear spaces. Soviet Math. Dokl. (1968), 61-63

[2] Dubinsky, E., Ramanujan, M. S. On λ-nuclearity Mem. of Amer. Math. Soc. No. 128 (1972)

[3] Köthe G. Stark nukleare Folgenräume J. Fac. Sci. Univ. Tokyo, I, 17, (1970) 291-296

[4] Martineau, A. Sur une propriété universelle de l'espace des distributions de L. Schwartz. C. R. Acad. Sci. Paris 259 (1964) 3162-3164

[5] Ramanujan, M. S., Power Series Spaces $\lambda(\alpha)$ and associated $\lambda(\alpha)$-nuclearity. Math. Ann. 189, (1970) 161-168

[6] Terzioglu, T., Smooth sequence spaces and associated nuclearity. Proc. of the Amer. Math. Soc. 37, 2 (1973) 497-504

[7] Terzioglu, T., Die diametrale Dimension von lokalkonvexen Räumen. Collect. Math., 20, (1969) 49-99

[8] Terzioglu, T., Nuclear and strongly nuclear sequence spaces. Revue Fac. Sci. Univ. Istanbul. Serie A34, (1969) 1-5