Projective descriptions and embedding theorems for φ_d

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ABSTRACT. We give projective representations of φ which lead to obtain embedding theorems into products of some non locally convex spaces. We introduce infinitely many different topologies on φ_d, intermediate between the box and the inductive topology. We give projective representations for φ_d carrying those topologies and show that, contrarily to what happens for φ, the results for embedding φ_d into product spaces are strongly negative.

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Introduction.

It is known that a countable dimensional linear space φ endowed with the strongest locally convex topology is always a subspace of an I-fold product E^I whenever $\mathrm{card}(I) \geq 2^{\aleph_0}$ and E does not carry its weak topology. This result first proved in [6] by S. Saxon does not extend to non-countable dimensional spaces φ_d $(d > \aleph_0)$ and the investigation of embedding theorems for φ_d into products lead in a natural way to look into the projective representation of several locally convex and non locally convex topologies in φ_d .

In the first part of the present paper we are concerned with the projective structure of φ and we show that all kernel topologies induced by diagonal operators on ℓ_p $0 < p < \infty$ do coincide on φ . As a consequence we extend Saxon's theorem to many non-locally convex spaces, such as ℓ_p $0 < p < 1$, Orlicz spaces, etc.

The second part aims to describe in what extent the above results remain valid for φ_d when d is uncountable. We show that the kernel topologies τ_p induced by diagonal operators on $\ell_p(I)$ $(\mathrm{card}\,(I) = d$, $0 < p \leq \infty)$ are all different for different p and we obtain an explicit projective representation of them. This family of kernel topologies τ_p carry a natural order depending on p and it include the box topology and the finest locally p-convex topology if $0 < p \leq 1$.

Concerning embedding properties of $[\varphi_d, \tau_p]$ into product spaces, we prove that they depend closely on p and d; roughly speaking, we show that $[\varphi_d, \tau_p]$ is a subspace of a large product of $l_q(I)$ if and only if $p = q$ and card $I \geq d$.

We would like finally to point out that the calculus of the associated Banach spaces of φ would also follows from its nuclearity and |7,8|, but those proofs do not provide the form of te linking maps (and are quite harder). On the other hand, they cannot apply to φ_d, since this space is not nuclear with any of the topologies τ_p (Prop. 3) - when $p = 1$ this is in [2, 10.4.2], and a similar argument serves for $p = \infty$ -.

1. Notations and Terminology

For the general terminology on topological vector spaces we refer throughout to [2] and [3].

$\mathbb{K}$ denotes the real or complex scalar field an $\mathbb{D}$ the closed unit disk in $\mathbb{K}$. As usual, for any locally p-convex space over $\mathbb{K}$, $0 < p \leq 1$ (1-convex = convex) and any absolutely p-convex neighborhood U of 0 in E, we denote by E_U the quotient space of E modulus the largest subspace contained in U. If Φ_U is the quotient map then E_U is always considered topologized with the p-norm $\|\Phi_U x\|_U = q_U(x)$ (q_U = gauge of U). $\hat{E}_U$ is the p-Banach space completion of E_U. If $V \subset U$, T_{VU} : $\hat{E}_V \longrightarrow \hat{E}_U$ is the extension to the completions of the canonical linking map $\Phi_V(x) \longmapsto \Phi_U(x)$, $x \in E$.

If I is a set of cardinality d, then φ_d is the space direct sum $\bigoplus_I \mathbb{K}$; when d is countable we simply write φ.

The so-called box topology on φ_d has a system of 0-neighborhoods formed by the sets $\bigoplus_I D_i$, where D_i are 0-neighborhoods in $\mathbb{K}$. It is also well know that for $0 < p \leq 1$ the finest locally p-convex topology on φ_d is given as an inductive topology (namely the finest locally p-convex topology making continuous all the inclusions $\mathbb{K} \hookrightarrow \bigoplus_I \mathbb{K}$).

2. Results for φ.

In [6], Saxon proved the following result: "Let E be a locally convex space. Then φ is a subspace of any product E^I, $\mathrm{card}(I) \geq 2^{\chi_o}$, if and only if E does not carry the weak topology".

This result relies upon the locally convex structure of E, rather than that of the φ. In [1], a simple proof using polarity arguments shows that a fundamental system of neighborhoods can be found in φ such that the associated Banach spaces are isometric to c_o.

We next give a further insight into the structure of φ. To begin with we fix some notations that will remain valid all throughout the paper.

For $0 < p \leq \infty$ we denote $\ell_p^+ = \{ \sigma = (\sigma_n) \in \ell_p ; \sigma_n > 0$ for all $n \in \mathbb{N} \}$ and we define the usual order $\sigma \leq \eta$ iff $\eta_n \leq K\sigma_n$ for all $n \in \mathbb{N}$ and for some constant $K > 0$. With respect to this order the subset ℓ_p^+ of ℓ_∞^+ is cofinal in ℓ_∞^+ whatever $0 < p < \infty$ (also any normal sequence space in ℓ_p^+ is cofinal in ℓ_p^+, $0 < p \leq \infty$). For every $\sigma = (\sigma_n) \in \ell_\infty^+$ we denote by $D_\sigma : (\xi_n) \longmapsto (\sigma_n \xi_n)$ the diagonal operator acting between appropiate sequence spaces. For every $\sigma \in \ell_\infty^+$ and $0 < p < \infty$ we define $E_\sigma^p = \ell_p$ and $E_\sigma = c_o$. If $\sigma \leq \eta$ we consider the diagonal operator $D_{\sigma^{-1}\eta}$ (with $\sigma^{-1}\eta = (\sigma_n^{-1}\eta_n)_{n \in \mathbb{N}}$) defined in ℓ_p or in c_o. The families $[E_\sigma^p , D_{\sigma^{-1}\eta}]_{\sigma \in \ell_\infty^+ , \eta \geq \sigma}$ and $[E_\sigma , D_{\sigma^{-1}\eta}]_{\sigma \in \ell_\infty^+ , \eta \geq \sigma}$

are projective system of topological linear spaces. With obvious notations we will denote by $\operatorname{proj}_{\sigma\in\ell_\infty^+} \ell_p$ and $\operatorname{proj}_{\sigma\in\ell_\infty^+} c_o$ the respective projective limits of the above projective systems.

Theorem 1.

If τ_b is the box topology in φ we have

$$[\varphi,\tau_b] \simeq \operatorname{proj}_{\sigma\in\ell_\infty^+} c_o$$

and for every $0 < p < \infty$

$$[\varphi,\tau_b] \simeq \operatorname{proj}_{\sigma\in\ell_\infty^+} \ell_p$$

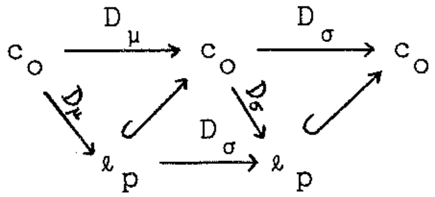
Proof. Let $V = \bigoplus_{n\in\mathbb{N}} \sigma_n \mathbb{D}$ $(:= (\prod_{n\in\mathbb{N}} \sigma_n \mathbb{D}) \cap \varphi)$ be an arbitrary τ_b-neighborhood of 0 in φ. The gauge q_V of V is clearly the norm

$$q_V(x) = \sup_{n\in\mathbb{N}} |\sigma_n^{-1} x_n| \qquad x = (x_n) \in \varphi$$

and therefore the associated Banach space $\widehat{\varphi}_V$ is isometric to c_o. If $U = \bigoplus_{n\in\mathbb{N}} \eta_n \mathbb{D}$, $0 < \eta_n \leq \sigma_n$ for all $n \in \mathbb{N}$, an standard extension argument yields that the linking map $T_{UV}: \widehat{\varphi}_U \longrightarrow \widehat{\varphi}_V$ is the diagonal operator $D_{\sigma^{-1}\eta}$ in c_o. Therefore

$$[\varphi,\tau_b] = \operatorname{proj}_V \widehat{\varphi}_V \simeq \operatorname{proj}_{\sigma\in\ell_\infty^+} c_o$$

and the first part is proved. For the second one let us recall that ℓ_p^+ is cofinal in ℓ_∞^+ for any $0 < p < \infty$ and consequently $\operatorname{proj}_{\sigma\in\ell_\infty^+} c_o = \operatorname{proj}_{\sigma\in\ell_p^+} c_o$. But if $\sigma, \mu \in \ell_p^+$, the following diagram is well defined and commutes



Therefore $\operatorname{proj}_{\sigma\in\ell_p^+} c_o = \operatorname{proj}_{\sigma\in\ell_p^+} \ell_p = \operatorname{proj}_{\sigma\in\ell_\infty^+} \ell_p$, the last equality being true again because ℓ_p^+ is cofinal in ℓ_∞^+.

Let us consider on ϕ the linear topologies τ_p, $0 < p \leq \infty$ defined by the system of norms (p-norms if $p < 1$)

$$q_{p\sigma}(x) = \left[\sum_{n=1}^{\infty} |\sigma_n^{-1}\xi_n|^p\right]^{1/p} \qquad x = (\xi_n) \in \phi$$

if $p < \infty$ and

$$q_{\infty\sigma}(x) = \sup_{n\in\mathbb{N}} |\sigma_n^{-1}\xi_n| \qquad x = (\xi_n) \in \phi$$

where $\sigma = (\sigma_n)$ ranges over ℓ_∞^+ .

If $V_{p\sigma}$ is the closed unit ball of the norm or p-norm $q_{p\sigma}$ $(0 < p \leq \infty)$, it is clear that $\widehat{\phi_{V_{p\sigma}}} \simeq \ell_p$ if $p < \infty$ and $\widehat{\phi_{V_{\infty\sigma}}} \simeq c_o$ whatever the sequence $\sigma \in \ell_\infty^+$. Furthermore if $V_{p\gamma} \subset V_{p\sigma}$ (i.e. if $0 < \gamma_n \leq \sigma_n$ for all $n \in \mathbb{N}$) , the linking map $T_{V_{p\gamma}V_{p\sigma}} : \widehat{\phi_{V_{p\gamma}}} \longrightarrow \widehat{\phi_{V_{p\sigma}}}$ is the diagonal map $D_{(\sigma_n^{-1}\gamma_n)}$ in ℓ_p (if $p < \infty$) or in c_o (if

$p = \infty$).

Therefore, by its very definition the topologies τ_p have the following projective representation

$$[\varphi, \tau_p] = \operatorname*{proj}_{\sigma \in \ell_\infty^+} \ell_p \qquad (0 < p < \infty)$$

$$[\varphi, \tau_\infty] = \operatorname*{proj}_{\sigma \in \ell_\infty^+} c_0$$

and the Theorem 1 can be reformulated as follows:

Theorem 1a.

The box topology τ_b in φ is equal to any of the topologies τ_p, $0 < p \leq \infty$.

In the following theorem we discuss an analogous description of the finest locally p-convex topology $(0 < p \leq 1)$ of φ in terms of τ_p .

Theorem 2.

For every $0 < p \leq 1$, the finest locally p-convex topology on φ is equal to τ_p .

Proof. Let $\mathcal{U}$ be the basis of zero neighborhoods in φ for the finest locally p-convex topology formed by all the sets $W = \Gamma_p(\bigcup_{n \in \mathbb{N}} \sigma_n \mathbb{D})$ where $\sigma = (\sigma_n)$ ranges over ℓ_∞^+ (here Γ_p stands for "absolutely p-convex cover"). Let us denote by q_W the p-norm gauge of W and by B_p the closed unit ball of the p-norm $\| \ \|_p$ of ℓ_p . For the diagonal injection $\Phi_{\sigma^{-1}} : [\varphi, q_W] \longrightarrow [\ell_p, \| \ \|_p]$,

$(x_n) \longmapsto (\sigma_n^{-1} x_n)$ we have clearly $\Phi_{\sigma^{-1}}(W) \subset B_p \cap \varphi$. Conversely if $\eta = (\eta_n) \in B_p \cap \varphi$, then all but finitely many η_n are zero and $\sum_n |\eta_n|^p \leq 1$. It follows that $(\eta_n \sigma_n) \in W$ because W is absolutely p-convex and furthermore $\Phi_{\sigma^{-1}}((\eta_n \sigma_n)) = (\eta_n)$. Thus $\Phi_{\sigma^{-1}}(W) = B_p \cap \varphi$ and $\Phi_{\sigma^{-1}}$ is a topological isomorphism onto a dense subspace of ℓ_p. We thus have $\widehat{\varphi}_W = \widehat{[\varphi, q_W]} \simeq \ell_p$. If $V = \Gamma_p(\bigcup_{n \in \mathbb{N}} \eta_n \mathbb{D})$ is another neighborhood in φ with $\eta_n \leq \sigma_n$ for all $n \in \mathbb{N}$, then the linking map $T_{VW} : \widehat{\varphi}_V \longrightarrow \widehat{\varphi}_W$ is the diagonal $D_{\sigma^{-1}\eta}$ on ℓ_p obtained extending by density the diagonal $D_{\sigma^{-1}\eta} : [\varphi, q_V] \longrightarrow [\varphi, q_W]$. Since $\sigma^{-1}\eta \in \ell_\infty^+$ it follows that the finest locally p-convex topology on φ has the same projective representation as τ_p. ∎

Remark. The Theorems 1a and 2 supply a new proof of the well known fact that on φ coincide the box topology and the finest locally convex topology ([2], 4.1.4.), and we will not distinguish them in what follows. This is not by far true for the non countable case as we will later see.

We recall that a topological linear space $[E, \tau]$ is called "pseudo-convex" if there exist a basis of zero-neighborhoods (U_α) for τ and a family (r_α) of numbers in $(0,1]$ such that U_α is absolutely r_α-convex for every α.

Theorem 3

Let λ be a pseudo-convex topological vector space of scalar sequences, such that $\varphi \subsetneq \lambda$ and λ is continuously contained in ℓ_∞. Let us assume that λ is normal and possesses a basis $\mathcal{U}$ of 0-neighborhoods satisfying the following condition.

(c) For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that if $|x_n| \leq |y_n|$ for all $n \in \mathbb{N}$ then $q_U((x_n)) \leq q_V((y_n))$.

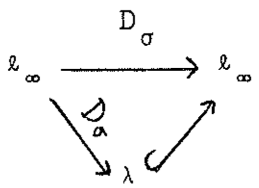
Then φ is a subspace of any product λ^J with card $(J) \geq 2^{\aleph_0}$.

Proof. Take $\sigma \in \lambda$ with $\sigma_n > 0$ for all $n \in \mathbb{N}$ (we exclude the interestless spaces λ such that $\lambda^+ = \emptyset$). The diagonal operator $D_\sigma : c_o \longrightarrow c_o$ is extended to a diagonal operator $D_\sigma : \ell_\infty \longrightarrow \ell_\infty$ and by normality of λ, $D_\sigma(\ell_\infty) \subset \lambda$.

By property (c) $D_\sigma : \ell_\infty \longrightarrow \lambda$ is continuous. Indeed, for every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that

$$q_U(D_\sigma((\xi_n))) = q_U((\sigma_n \xi_n)) \leq q_V(\sigma) \; \| (\xi_n) \|_\infty$$

and the criterion [2] Th. 6.5.4. applies. Since, by hypothesis the canonical inclusion $\lambda \hookrightarrow \ell_\infty$ is also continuous we get the following continuous factorization



Since λ is normal, λ^+ is cofinal in ℓ_∞^+ and, thus, using the Theorem 1a, we have $\varphi \simeq \underset{\sigma \in \ell_\infty^+}{\text{proj}}\, c_0 \subset \underset{\sigma \in \ell_\infty^+}{\text{proj}}\, \ell_\infty = \underset{\sigma \in \lambda^+}{\text{proj}}\, \ell_\infty = \underset{\sigma \in \lambda^+}{\text{proj}}\, \lambda \subset \lambda^J$ where the cardinality of J is the cardinality of a neighborhood basis of φ, i.e. card $(J) \geq 2^{\aleph_0}$ because φ is not metrizable. ∎

Applications of this result not covered by [6] are

1. $\lambda = \ell_p \quad , \quad 0 < p < 1$
2. $\lambda = \bigcap_{0 < p < 1} \ell_p$ endowed with the projective induced topology. This space is not even p-convex for any $p < 1$ but only pseudo-convex ([2] 6.10.G.)
3. $\lambda = \Lambda_\phi^p(P) \quad , \quad 0 < p < 1$, non locally convex power series sequence spaces (see [5]).
4. $\lambda = \ell_\rho$ Orlicz sequence spaces not locally convex |9|.
5. Any topological linear space containing subspaces as above, ej. $E = \mathcal{L}_p(X, \Omega, \mu) \quad 0 < p < 1$, X infinite.

3. Results for φ_d

Let us consider now the space φ_d where $d = \mathrm{card}(I) \geq 2^{\aleph_0}$. We fix in the sequel such an index set I. We define on φ_d the topologies τ_p, $0 < p \leq \infty$ by means of the norms (p-norms if $p < 1$)

$$q_{p\sigma}(x) = \left[\sum_{i \in I} |\sigma_i^{-1} \xi_i|^p\right]^{1/p} \qquad x = (\xi_i)_{i \in I} \in \varphi_d$$

if $p < \infty$ and

$$q_{\infty\sigma}(x) = \sup_{i \in I} |\sigma_i^{-1} \xi_i| \qquad x = (\xi_i)_{i \in I} \in \varphi_d$$

where $\sigma = (\sigma_i)$ ranges over $\ell_\infty(I)^+$.

As in the countable case we have

$$[\varphi_d, \tau_p] = \mathrm{proj}_{\sigma \in \ell_\infty(I)^+} \ell_p(I)$$

$$[\varphi_d, \tau_\infty] = \mathrm{proj}_{\sigma \in \ell_\infty(I)^+} c_o(I)$$

The techniques developed in Theorems 1 and 2 generalize to the non-countable case. We thus omit the proof of the following.

Theorem 4.

Let $0 < p \leq 1$ be; τ_p is the finest locally p-convex topology on φ_d. τ_∞ is the box-topology on φ_d.

In contrast with the countable case we have however

Proposition 1.

If $0< s< r< 1< q< p<\infty$, then $\tau_\infty \leq \tau_p \leq \tau_q \leq \tau_1 \leq \tau_r \leq \tau_s$. Moreover all these topologies are different on φ_d.

Proof. Let $0 < q < p \leq \infty$ be. From the relation

$$q_{p\sigma}(x) \leq q_{q\sigma}(x) \qquad x \in \varphi_d \qquad \sigma \in \ell_\infty(I)^+$$

we deduce $\tau_p \leq \tau_q$. In order to prove that $\tau_p \neq \tau_q$ we recall the following simple fact: if $q < p$ and $\sigma \in \ell_\infty(I)^+$, then the diagonal map D_σ cannot carry $\ell_p(I)$ into $\ell_q(I)$ unless I be countable ($\ell_p(I)$ is understood to be $c_0(I)$ if $p = \infty$) .

Let us observe as well that the topology τ_p in φ_d is the Kernel topology corresponding to the family of diagonal maps $D_{\sigma^{-1}} : \varphi_d \longrightarrow \ell_p(I)$, $\sigma \in \ell_\infty^+(I)$. Therefore the equality $\tau_p = \tau_q$ in φ_d would imply, by density, the continuity of some $D_{\sigma^{-1}}: \ell_p(I) \longrightarrow \ell_q(I)$, $\sigma \in \ell_\infty(I)^+$. But this is impossible because I is uncountable. ∎

We study now some results concerning embedding of $[\varphi_d , \tau_p]$, $0 < p \leq \infty$, into large products.

Proposition 2.

Let E be an infinite dimensional locally convex space. If $[\varphi_d , \tau_p]$ is a subspace of some product E^J , then E has a basis of zero neighborhoods $\mathcal{U}$ such that $\dim (\hat{E}_U) \geq d$ for every $U \in \mathcal{U}$.

Proof. We will suposse that E is a Banach space X. Minor changes will provide the general case. Take a continuous norm q in $[\varphi_d, \tau_p]$ (for example any $q_{p\sigma}$) and let U be its closed unit ball. From $[\varphi_d, \tau_p] \subset X^J$ we can determine a neighborhood B of 0 in X such that

$$V := (\overbrace{B \times \cdots\cdots\cdots \times B}^{n \text{ times}} \times X^J) \cap \varphi_d \subset U$$

it follows that $(\varphi_d)_V$ is (algebraically) a subspace of X^n. Since q is norm we finally conclude

$$d = \dim \varphi_d = \dim (\varphi_d)_V \leq \dim X^n = \dim X$$

∎

In the following corollaries we deduce some facts contrasting strongly with the situation described in the theorem of [6] quoted at the begining of the Section 2.

Corollary 1. Let J and Λ be index sets such that $\mathrm{card}(J) \geq 2^d$. Then

a) $[\varphi_d, \tau_p]$, $0 < p < \infty$, is a subspace of $\ell_p(\Lambda)^J$ if and only if $\mathrm{card}(\Lambda) \geq d$.

b) $[\varphi_d, \tau_\infty]$ is a subspace of $c_0(\Lambda)^J$ if and only if $\mathrm{card}(\Lambda) \geq d$.

Proposition 3. If E is a Schwartz space, then no product of E contains $[\varphi_d, \tau_p]$ whatever $0 < p \leq \infty$.

Proof. Since E is a Schwartz space, the associated Banach spaces $\widehat{E}_U$ are separable.

On the other hand,

$[\varphi_d , \tau_p]$, $0 < p \leq \infty$, is not a Schwartz space because the associated Banach spaces of $[\varphi_d , \tau_p]$ are isomorphic to the (non-separable) spaces $\ell_p(I)$ if $0 < p < \infty$ or $c_o(I)$ if $p = \infty$.

Lemma.

A diagonal operator $D_\sigma : \ell_p(I) \longrightarrow \ell_p(I)$, $0 < p \leq \infty$, (if $p = \infty$ one must understand $c_o(I)$), $\sigma \in \ell_\infty(I)^+$, cannot be continuously factorized through $\ell_q(I)$ if $q \neq p$.

Proof. An operator $A : \ell_p(I) \longrightarrow \ell_q(I)$ is represented by a "matrix" $(a_{ij})_{(ij) \in IxI}$ in the following sense:

$$\text{if} \quad A((x_j)) = (y_i) \qquad \text{then} \quad y_i = \sum_{j \in I} a_{ij} x_j$$

Suppose $p > q$. It is not hard to check that the ma trix of the operator A satisfies

(1) For each $j \in J$ the set of indexes $i \in I$ such that $a_{ij} \neq 0$ is countable.

(2) For all, but finitely many $i \in I$, the set of indexes $j \in I$ such that $a_{ij} \neq 0$ is countable (there is no loss of generality assuming that this condition holds for every $i \in I$) .

We then claim that A has the following property:

(*) The set $I_o = \{(i,j) \in IxI \ ; \ a_{ij} \neq 0\}$ is countable.

Should this not be the case, and assuming $a_{ij} > 0$

for uncountable (i,j), then there exists $\varepsilon > 0$ such that for an uncountable set $M \subset I \times I$, $a_{ij} > \varepsilon$ if $(i,j) \in M$

Appealing to (1) and (2), we deduce that the indexes in M need to be scattered through uncountably many rows and columns of $I \times I$. Pick then a countable set $J \subset M$ such that for all $a, b \in J$

$$\Pi_1(a) = \Pi_1(b) \Rightarrow a = b$$

and

$$\Pi_2(a) = \Pi_2(b) \Rightarrow a = b$$

(here Π_1 and Π_2 are the respective projections of $I \times I$ into I).

Choose next a $z = (z_j) \in \ell_p(I) \setminus \ell_q(I)$ with $z_j > 0$ iff $j \in \Pi_2(J)$. If $Az = y$, then for each pair $(i,j) \in J$ we have

$$y_i = \sum_k a_{ik} z_k > \varepsilon z_j$$

and thus

$$\sum_{i \in I} |y_i|^q \geq \sum_{i \in \Pi_1(J)} |y_i|^q > \varepsilon^q \sum_{j \in \Pi_2(J)} |z_j|^q =$$

$$= \varepsilon^q \sum_{i \in I} |z_j|^q = +\infty$$

This contradiction proves (*).

Suppose now D_σ factorizes as $D_\sigma = B \circ A$ with $A : \ell_p(I) \longrightarrow \ell_q(I)$ and $B : \ell_q(I) \longrightarrow \ell_p(I)$. By (*) we can choose $j \in I \setminus \Pi_2(I_0)$ and define $x^{(j)} \in \ell_p(I)$ as $x_k^{(j)} = \delta_{kj}$, $k \in I$. We then have $D_\sigma(x^{(j)}) \neq 0$ (note its j-th component is $\sigma_j \neq 0$) but if $y = A(x^{(j)})$ then $y_i = \sum_{k \in I} a_{ik} x_k^{(j)} = a_{ij} = 0$ for all $i \in I$ because $j \notin \Pi_2(I_0)$. Therefore $A(x^{(j)}) = 0$ which is impossible. That proves our Lemma in the case $p > q$. If $p < q$ we then observe that B has also the corresponding property (*) and therefore $B(\ell_q(I)) \subset \ell_p(\mathbb{N})$ ($\mathbb{N}$ denotes here, of course, a countable subset of I) . But we get again a contradiction because the image of D_σ cannot lay on $\ell_p(\mathbb{N})$

■

Remark. Let us note that if $p > q$ in the previous Lemma, D_σ cannot be even "subfactorized" through $\ell_q(I)$ in the sense that there is no subspace $Z \subset \ell_q(I)$ and operator $B \in \mathcal{L}(Z, \ell_p(I))$ such that $A(\ell_p(I)) \subset Z$ and $D_\sigma = B \circ A$

It is also worth noticing the remarkable contrast between this Lemma and the factorization argument used in the proof of the Theorem 1 for the countable case.

Theorem 5.

Let $d \geq 2^{\aleph_0}$ and $0 < q < p \leq \infty$ be. Then $[\psi_d , \tau_p]$ is not a subspace of any product $\ell_q(I)^J$.

Proof. Let us first note that any product $(\ell_q(I))^J$ has a basis $\mathcal{U}$ of zero-negihborhoods such that for $U \in \mathcal{U}$ the associated q-Banach space $(\widehat{\ell_q(I)^J})_U$ is topologically isomorphic to $\ell_q(I)$. If $[\varphi_d, \tau_p]$ were a subspace of $\ell_q(I)^J$, then, recalling the projective representation of τ_p at the begining of this Section, we would obtain a factorization of diagonal operators $D_\sigma : \ell_p(I) \longrightarrow \ell_p(I)$ $\sigma \in \ell_\infty(I)^+$ through subspaces $(\widehat{\varphi_d)_{U \cap \varphi_d}}$ of $\ell_q(I)$, $U \in \mathcal{U}$. Such a subfactorization of D_σ through $\ell_q(I)$ is not possible according to the previous Lemma and its subsequent remark.

As a complement of the Theorem 5, we will next prove that its validity can be extended to the following more trivial setting:

If $0 < p < q < 1$ (resp. $0 < p < 1 \leq q \leq \infty$) then $[\varphi_d, \tau_p]$ cannot be a subspace of $\ell_q(I)^J$. Otherwise, by [2] 6.6.3., τ_p would be a locally q-convex topology (resp. a locally convex topology). This, in turn, implies, by the Theorem 4, that $\tau_p = \tau_q$ (resp. $\tau_p = \tau_1$) contradicting the Proposition 1.

Conjecture. We feel strongly toward the following refinement of the present paper:

Theorem 5 should be true for all $p \neq q$. This would follow from the preceding Lemma as far as this Lemma could be proved for subfactorizations even when $p < q$.

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