UNIVERSIDAD DE EXTREMADURA

TESIS DOCTORAL

Tensores de divergencia nula asociados a una métrica

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Introduction

Let (X,g) be a Lorentz manifold or, more generally, a pseudo-Riemannian manifold. This memory discusses 2-covariant tensors on X that are constructed out of the metric g and another tensor field ω, and which are divergence-free. These tensors are important because any matter-energy distribution on spacetime is represented by one of these tensor fields, where the divergence-free condition expresses an infinitesimal conservation law.

There exist plenty of examples: the Euler-Lagrange tensor associated to any variational problem defined in the bundle of metrics, whose lagrangian function is a scalar invariant of metrics, is automatically divergence-free. This fact is a particular case of the second Noether's theorem.

The study of this memory is focused on the following examples:

• The Einstein tensor, that represents the energy content of a relativistic spacetime.

• The energy tensor associated to an electromagnetic field, and its generalization to the electromagnetism of charged p-branes.

• The Lovelock tensors, that appear in dimension greater than 4.

The main results contained in this thesis consist of different characterizations of each of these tensors.

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Preliminaries

The divergence-free tensors under consideration are *naturally constructed* out of a metric. But what does it exactly mean that a tensor is constructed out of another tensor, and what does the assumption of naturalness signify? This memory begins with two preliminary chapters, where we address these questions. Our exposition differs from those encountered in the literature, so we include complete proofs.

Tensors are sections of a bundle, so that, more generally, let us consider two bundles, $F \to X$ and $\overline{F} \to X$, and their sheaves of smooth sections $\mathscr{F}$ and $\overline{\mathscr{F}}$ over X. Let us suppose that, for each local section s of F, we have a procedure to assign (or "construct") another local section $\varphi(s)$ of $\overline{F}$, satisfying the following reasonable assumptions:

-Local character: If the section s is defined on an open set $U \subseteq X$, then $\varphi(s)$ is also defined on U, and the map φ is compatible with restrictions: for each open set $V \subseteq U$, it holds $\varphi(s_{|V}) = \varphi(s)_{|V}$. In other words, this amounts to saying that φ is a morphism of sheaves $\varphi\colon \mathscr{F} \to \overline{\mathscr{F}}$.

-Regularity: If $\{s_t\}_{t\in T}$ is a smooth family of sections of F, parametrized by a smooth manifold T, then $\{\varphi(s_t)\}_{t\in T}$ is also a smooth family of sections of $\overline{F}$.

The first Chapter is devoted to give a proof of the following result:

Theorem (Slovák, [44]): *Any regular morphism of sheaves $\varphi\colon \mathscr{F} \to \overline{\mathscr{F}}$ is a differential operator.*

This theorem responds the question of what should we understand by a tensor $\varphi(s)$ constructed out of another tensor s. The answer is that such a construction has to be defined by a differential operator φ between the involved bundles. As an example, the map that assigns to each metric g its corresponding Einstein tensor G defines a differential operator of order two, $g \mapsto G$, between the bundle of metrics and the bundle of 2-tensors.

A differential operator $\varphi\colon \mathscr{F} \to \overline{\mathscr{F}}$ is *natural* if its definition does not depend on choices of local coordinates on X. As a change of coordinates is, essentially, the same thing as a diffeomorphism, we can specify the previous definition as follows: a differential operator $\varphi\colon \mathscr{F} \to \overline{\mathscr{F}}$ is natural if it commutes with the action of diffeomorphisms on the base manifold X; that is, if for any diffeomorphism $\tau: U \to V$ between open sets of X, and for any section $s \in \mathscr{F}(U)$, it holds $\varphi(\tau_*(s)) = \tau_*(\varphi(s))$.

The theory of natural bundles and natural operators is classical; in the second Chapter we prove some of its basic results that will be used later on. At this stage, our presentation differs from the standard reference - the book by Kolár-Michor-Slovák ([24]) - but is equivalent to it.

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Chapter 3: Homogeneous tensors

There is an elementary observation that plays a key role in the results of this memory: if the metric g on a spacetime (X,g) is changed by a proportional one, $\lambda^2 g$, with $\lambda \in \mathbb{R}^+$, then the geometry of space-time remains unaltered, whereas the measures of time and length are modified by a constant factor. The proper time of a timelike trajectory (i.e., its arc-length) as well as spatial lengths (of spatial-like vectors) get multiplied by λ, while the Levi-Civita connection remains unaffected. That is, replacing the metric g by a proportional one $\lambda^2 g$ amounts to modifying the units of time and length, without varying the geometry of space-time.

In fact, the underlying question is that the "correct" definition of a pseudo-Riemannian structure on a manifold should not consist on a metric g, but on a one-dimensional subspace $< g >_{\mathbb{R}^+}$ of metrics, without specifying a particular generator. Therefore, any reasonable tensor $T(g)$ constructed out of a metric g, should not be altered (up to a factor) when the metric g is substituted by $\lambda^2 g$, with $\lambda \in \mathbb{R}^+$; that is, it should satisfy a formula of the type

$$T(\lambda^2 g) = \lambda^w T(g)$$

for all $\lambda \in \mathbb{R}^+$, where the exponent w indicates the kind of dependence of T on the units of time-length. Tensors fulfilling such a formula are called *homogeneous*, and the exponent w is called the *weight* of the tensor.

Our conviction on this notion could be summarized in the following:

Principle of homogeneity: *In Differential Geometry, any "relevant" tensor associated to a metric g must be natural and homogeneous.*

This principle could be understood as a differential geometric version of the dimensional analysis in Physics. Examples of homogeneous tensors include the curvature tensor, R, or the Ricci tensor, Ric, which are homogeneous of weight zero. The scalar curvature, r, is homogeneous of weight -2, whereas the volume form ω_X is homogeneous of weight the dimension of X. The principle of homogeneity may also be applied to non-tensorial objects; v.gr., the Levi-Civita connection satisfies $\nabla_{\lambda^2 g} = \nabla_g$ and D'Alembert's differential operator $\Box_{\lambda^2 g} = \lambda^{-2} \Box_g$.

On the other hand, this homogeneity condition has already been considered in the literature. For instance, let us highlight the characterization of the Levi-Civita connection, due to Epstein ([13]), as the only linear connection naturally associated to a metric that satisfies $\nabla_{\lambda^2 g} = \nabla_g$. Another important result, due to Gilkey ([17]), characterizes the Pontryagin forms as the only differential forms naturally constructed out of a metric g, that are homogeneous of weight zero.

The main result in Chapter 3 is a formula that allows to determine all the p-tensors $T(g,\omega)$, naturally associated to a metric g and to a k-covariant tensor ω, satisfying an homogeneity condition $T(\lambda^2 g, \lambda^a \omega) = \lambda^w T(g,\omega)$, for certain $a, w \in \mathbb{R}$. The formula, inspired by similar results due to Stredder ([47]) and Slovák ([45]), reduces the computation of such tensors to a problem of invariants of the orthogonal group, which in turn may by tackled using classical invariant theory.

Our statement (Theorem 3.4.2) is somewhat technical to be introduced here, so let us simply point out its main consequences as follows:

Theorem: *The vector space of p-tensors $T(g,\omega)$ naturally associated to a metric g and a k-covariant tensor ω, and satisfying the homogeneity condition $T(\lambda^2 g, \lambda^a \omega) = \lambda^w T(g,\omega)$, is finite dimensional.*

Moreover, it is possible to explicitly describe a system of generators of such space. These generators are defined using tensor products and contractions of the tensors $g, R, \nabla R, \nabla^2 R, \dots$ and $\omega, \nabla\omega, \nabla^2\omega, \dots$.

Chapter 4: Characterization of the Einstein tensor

In General Relativity, spacetime is conceived as a Lorentz manifold (X,g) of dimension 4 and gravity is interpreted as a manifestation of the curvature of such manifold. To formulate the field equation of this theory, Einstein had in mind that, in the Newtonian theory, the mass distribution is determined by the gravitational force via the Poisson equation. As in the relativistic theory the gravitational force is substituted by the geometry of the manifold, the field equation of the theory should determine the mass distribution out of the metric g; that is, it should be an equation of the form:

$$G(g) = \mu\, T$$

where $G(g)$ is a 2-covariant tensor constructed out of g, T is the energy-momentum 2-covariant tensor of the matter distribution and $\mu \in \mathbb{R}$ is a constant depending on the fixed units.

But the matter tensor T satisfies the condition $\operatorname{div} T = 0$, that encodes the infinitesimal conservation of impulse, so the unknown tensor $G(g)$ has also to be divergence-free. At the time, the only 2-tensor known to be divergence-free was $Ric - \frac{r}{2}g$, so Einstein deduced that the field equation of General Relativity should be:

$$Ric - \frac{r}{2}g = \mu T \ .$$

The choice of $G(g) = Ric - \frac{r}{2}g$ is motivated by the following result, first published by H. Vermeil ([51]), and independently proved by E. Cartan ([9]):

Theorem (Vermeil-Cartan): *Up to a constant factor and the addition of a cosmological term Λg, the Einstein tensor $Ric - \frac{r}{2}g$ is the only symmetric, divergence-free, natural 2-tensor that is second-order (in the derivatives of the coefficients g_{ij} of the metric) and that is linear on these second derivatives.*

Fifty years later, this result was improved by D. Lovelock ([31]), who proved that, in dimension 4, the hypotheses of symmetry and linearity on the second derivatives are superfluous.

In this memory we provide a new characterization of the Einstein tensor, suppressing the limitation on the order of derivatives used to define it, but requiring instead a condition of independence on the units of scale:

Theorem 4.2.2: *Up to a constant factor, the Einstein tensor $G(g) = Ric - \frac{r}{2}g$ is the only divergence-free, natural 2-tensor that is independent of the unit of scale: $G(\lambda^2 g) = G(g)$.*

Even more, any divergence-free, natural 2-tensor that is homogeneous of weight $w > -2$ is a constant multiple of the Einstein tensor, if $w = 0$, or a constant multiple of the metric g, if $w = 2$.

In Newtonian gravitation, if we fix the unit of mass so that the constant of universal gravitation becomes 1, then the energy-momentum tensor T of the matter distribution does not depend on the time and length units. Hence, the same should occur in the relativistic case, that approximates the Newtonian theory: the mater tensor T has to be independent of the unit of time. Thus, the previous theorem proves that the Einstein tensor is, up to a constant factor, the only tensor naturally associated to the metric, that satisfies the two basic properties of the matter tensor: it is divergence-free and it is independent of the unit of scale. That is to say, our theorem provides a strong theoretical justification for the choice of the Einstein tensor in the field equation of General Relativity.

The application of dimensional analysis to the Einstein equation has a precedent in the work of Aldersley ([1]), but his formulation is rather obscure (see the comment in page 56). On the other hand, the proof of our theorem is quite simple: the condition of $G(g)$ being independent of the unit of scale is equivalent, in virtue of the general formula of Chapter 3, to saying that $G(g)$ only depends on second derivatives of g, and that it is linear on these second derivatives. Moreover, some standard arguments using invariant theory allow to show that $G(g)$ has to be symmetric, hence reducing the question to the classical result of Vermeil-Cartan.

It is worth pointing out that our characterization of the Einstein tensor also remains valid if we consider 2-tensors $T(g, \text{orient})$ naturally associated to a metric g and an orientation. The possibility of depending on the orientation is not vain, as illustrate, v.gr., the Cotton tensors. These tensors, that depend on the metric up to its third derivatives and on the orientation, are known for being the only natural 2-tensors that can be derived from a variational principle, but for which there not exist a natural lagrangian ([5]). The Cotton tensors are homogeneous of weight -2, so there is no contradiction with the above characterization.

The second part of the Chapter studies the field equation on a Weyl spacetime. Let us firstly recall that a *Weyl spacetime* consists on a smooth manifold X of dimension $1+n$, endowed with a symmetric linear connection ∇ and a conformal Lorentzian structure $[g]$, both of which are compatible in the sense that

$$\nabla g = \alpha \otimes g$$

for some 1-form α.

The 1-form α depends on the metric g representing the conformal structure. The 1-forms associated to different representatives differ on an exact form, so the 2-form $\omega = \mathrm{d}\alpha$ is intrinsic, and we call it the *Weyl form*.

From a purely mathematical point of view, the Weyl spacetime appears as a generalization of the Minkowski spacetime, even more faithful than that of a Lorentz spacetime. It is the Weyl geometry, not the Lorentz's, that possesses the same infinitesimal structure (at first order) than the Minkowski geometry.

Back in 1918, H. Weyl ([52]) formulated this geometry as an attempt to unify gravitation and electromagnetism, with the aim of interpreting the Weyl 2-form ω as the 2-form of the electromagnetic field. But, as already observed Einstein, this theory did not correspond with experimental facts; Weyl himself discarded it in favour of another model that gave birth to the modern gauge theories ([53]). In fact, the dimensional analysis already shows that, contrary to the 2-form of the electromagnetic field, the 2-form ω is independent of the units of scale, so that ω cannot be interpreted as the electromagnetic 2-form.

But the beauty of the Weyl geometry invites to study it as a pure theory of gravitation, where the Weyl 2-form is no longer interpreted as electromagnetism, but as mere curvature. In order to write down a field equation for this theory, we need to find all possible 2-tensors that are divergence-free.

We solve this question with the following:

Theorem 4.2.15: *The only 2-tensors, naturally associated to a Weyl geometry* $(\nabla, [g])$ *are the linear combinations of:*

$$Ric_\nabla \quad , \qquad rg \quad , \qquad \omega \, .$$

Up to a constant factor, the only divergence-free 2-tensor naturally associated to a Weyl geometry is

$$G := Ric_\nabla - \frac{r}{2}g + \frac{n}{4}\omega \, .$$

Nevertheless, this 2-tensor G is not symmetric: its skew-symmetric component is $-\frac{1}{4}\omega$. By reasons that are not clear to us, it is commonly accepted that the energy-momentum tensor T of the matter distribution should be symmetric. In such a case, a field equation $G = \mu T$ already implies $\omega = 0$, that amounts to saying that Weyl geometry is locally Lorentzian.

That is, we arrive to the conclusion that the exigence of the symmetry of T implies that is not possible, at least locally, to generalize General Relativity using a Weyl geometry.

Chapter 5: Characterization of the electromagnetic energy tensor

Let (X, g) be a spacetime, i.e., an oriented Lorentz manifold of dimension $1+n$. An electromagnetic field on it is represented by a 2-form F on X that satisfies the Maxwell equations,

$$\mathrm{d}F = 0 \qquad , \qquad \partial F = J^* \, ,$$

where J is the charge-current vector field.

The electromagnetic energy tensor T_{elm} corresponding to this field F is a 2-covariant tensor, locally defined on a chart as:

$$T_{ab} := -\left(F_a{}^i F_{bi} - \frac{1}{4} F^{ij} F_{ij} g_{ab} \right) \, . \tag{0.0.0.1}$$

Our purpose in the fifth Chapter is to prove a characterization of this tensor, motivated by physical grounds.

The reason for the introduction of this electromagnetic energy tensor is the necessity of preserving the principle of conservation of energy: in absence of electric charges, the energy-momentum tensor T_m of a fluid has to fulfil the condition:

$$\operatorname{div} T_m = 0 \, ,$$

that encodes the infinitesimal conservation of mass-energy and momentum. But in presence of electric charges in the fluid, movement becomes affected by the electromagnetic field, according to the Lorentz Force Law,

$$\operatorname{div} T_m = i_J F = i_{\partial F} F$$

that, in particular, would imply a violation of the conservation of energy. This can remedied assuming that, apart from the energy-momentum T_m of matter, the electromagnetic field itself possesses energy, T_{elm}, in such a way that the total energy is conserved:

$$\operatorname{div}(T_m + T_{\mathrm{elm}}) = 0 \ .$$

Therefore, we need to find a 2-tensor T_{elm}, associated to the electromagnetic field F, whose divergence cancels the divergence of the matter tensor T_m, that is, $\operatorname{div} T_{\mathrm{elm}} = -i_{\partial F} F$. Moreover, in order that the adding $T_m + T_{\mathrm{elm}}$ makes sense, both tensors should have the same dependence of the unit of time-length: as T_m is independent of the unit of time-length, the same should occur to T_{elm}. Lastly, it is sensible to assume that, wherever the electromagnetic field is null, there is no electromagnetic energy.

Our main theorem asserts that these three properties, that any reasonable definition of T_{elm} should hold, uniquely characterize it:

Theorem 5.2.5: *The energy tensor (0.0.0.1) is the only* 2*-tensor* $T = T(g,F)$ *naturally associated to a Lorentzian metric* g *and a closed* 2*-form* F*, satisfying the following properties:*

1. *T is independent of the unit of scale:* $T(\lambda^2 g, \lambda F) = T(g,F)\ , \quad \forall \lambda \in \mathbb{R}^+ \ .$
2. $\operatorname{div} T = -i_{\partial F} F$*,*
3. *At any point,* $F_x = 0 \ \Rightarrow \ T_x = 0$.

This problem of characterizing the electromagnetic energy tensor is also classical and has already been studied in the literature ([4], [22], [23], [32], [33]). The closest result to our statement is Kerrighan's ([23]), where the tensor $T(g,F)$ is assumed to be symmetric, defined for any pair (g,F) (not only when F is closed), and its coefficients are assumed to be functions of the coefficients of g and F (so tensors using higher derivatives of g and F are not considered). These restrictions are removed in our theorem where, instead, we require independence of the unit of scale.

The energy tensor (0.0.0.1) may be defined for k-forms of arbitrary order k, or even for more arbitrary tensors; these are the *super-energy tensors* introduced by Senovilla ([43]). The energy tensor of a k-form admits a similar characterization to that stated above for

2-forms, and this fact suggested us the idea of a possible physical interpretation for these energy tensors. In ([15]), M. Henneaux and C. Teitelboim introduced a generalized theory of electromagnetism where charged particles are not punctual, but have dimension p (hence called p-*branes*), and where the electromagnetic field is represented by a $(2+p)$-form. In this memory, we add two basic points to their formulation: the Lorentz Force Law, introducing the notion of acceleration of a p-brane, and the Einstein field equation. In order to write this field equation, we explain how the energy tensor of a $(2+p)$-form must be understood as the energy tensor of the electromagnetic field of this theory, thus obtaining a physical interpretation for these tensors.

Chapter 6: Divergence-free, second-order tensors

The final Chapter deals with the general problem of describing natural tensors that are divergence-free. Due to the complexity of the question, we restrict our attention to *second-order tensors*; i.e., tensors that are second-order in the coefficients g_{ij} of the metric. Apart from this, we consider an arbitrary number of indices and symmetries among them.

There exists abundant literature on this topic ([3], [12], [29], [30]). The major breakthrough in the area remains the work done by D. Lovelock regarding symmetric tensors with 2 indexes. In ([30]), he introduced a sequence of tensors $\{L_0,\dots,L_m\}$, where $2m \leq n-1$ and $n = \dim X$, for which he proved:

Theorem (Lovelock): *The tensors* $\{L_0,\dots,L_m\}$*, where* $2m \leq n-1$*, are a basis of the* $\mathbb{R}$*-vector space of second-order, natural* 2*-tensors that are symmetric and divergence-free.*

The first two of these *Lovelock tensors* are the metric, $L_0 = g$, and the Einstein tensor $L_1 = G$. Hence, if $n = 4$:

Corollary: *In dimension* $n = 4$*, the only second-order, natural* 2*-tensors that are symmetric and divergence-free are the* $\mathbb{R}$*-linear combinations of the metric* g *and the Einstein tensor* G*.*

These Lovelock tensors $L_0,\dots,L_k$ are closely related to the Chern-Gauss-Bonnet theorem ([19]) and have received much attention by the physical community (see [11], [20] and references therein).

Nevertheless, a similar analysis for tensors with different symmetries or with a higher number of indices revealed difficult ([8], [12]). Some progress has been made with the aid of computer programs ([7]), but there is still a lack of general results.

Our purpose in this Chapter is twofold: on the one hand, we aim to strengthen Lovelock's theorem, and, on the other, we try to simplify Lovelock's original proof.

As regards to the first intention, we obtain:

Theorem 6.4.3: *The hypothesis of symmetry in the Lovelock's theorem is superfluous.*

In other words, we prove that the tensors $\{L_0,\dots,L_m\}$, where $2m \leq n-1$, are a basis of the $\mathbb{R}$-vector space of second-order, natural 2-tensors that are divergence-free. This fact was already proved by Lovelock himself in dimension $n=4$, but the general case remained open.

Moreover, our methods also allow to produce new results for tensors with more than two indices. For instance:

Propositions 6.5.4: *There are no second-order, natural p-forms ($p \geq 1$) that are divergence-free, but the zero form.*

As concerns the clarifying of Lovelock's proof, we propose an alternative definition of the Lovelock tensors, that makes no use of coordinates, and that allows to immediately check their symmetry and the vanishing of their divergence.

Besides, the original proof due to Lovelock involved lengthy calculations using multi-indexes, that we have strongly simplified, using elements from the invariant theory of the orthogonal group, as well as elementary facts of graph theory.

a.m.D.g.

Chapter 1

Differential operators

Let $F \to X$ and $\overline{F} \to X$ be bundles over a smooth manifold X. Let ϕ be an "assignment" that maps smooth sections of F into smooth sections of $\overline{F}$.

Let us suppose ϕ satisfies the following conditions:

1. *Local character:* It defines a morphism of sheaves $\phi\colon \mathscr{F} \to \overline{\mathscr{F}}$ between the sheaf of smooth sections of F and that of $\overline{F}$. In particular, the value of $\phi(s)$ at a point $x \in X$ is determined by the value of s in a neighbourhood of x.

2. *Regularity:* If $\{s_t\}_{t\in T}$ is a smooth family of sections of F, then $\{\phi(s_t)\}_{t\in T}$ is a smooth family of sections of $\overline{F}$.

This Chapter is devoted to prove that any such a ϕ is indeed a *differential operator*: $\phi(s)(x)$ only depends on (a locally finite number of) derivatives of s at x.

The content of this Chapter is inspired in the work of Slovák ([24], [44]), although our presentation uses the language of sheaves and ringed spaces to clarify the exposition.

1.1 Whitney's Extension Theorem

In this Section we formulate the Whitney's Extension Theorem, and we prove some consequences that will be used in the proof of the Slovák's Theorem 1.3.4.

For any pair of multi-indexes,

$$I = (r_1, \dots, r_n)\ ,\ J = (\bar{r}_1, \dots, \bar{r}_n)\ \in \mathbb{Z}^+ \times \overset{n}{\dots} \times \mathbb{Z}^+$$

let us introduce the following notations:

$$|I| := r_1 + \ldots + r_n \quad , \quad I! := r_1! \ldots r_n! \quad , \quad I + J := (r_1 + \overline{r}_1, \ldots, r_n + \overline{r}_n) \, .$$

Some other standard multi-index notation will be used throughout this Section, without more explicit mention. As an example, the Taylor expansion of a smooth function f on $\mathbb{R}^n$ at a point $a \in \mathbb{R}^n$ is denoted:

$$\mathrm{T}_a f := \sum_I \frac{D_I f(a)}{I!} \, (x-a)^I \, .$$

Definition. A Taylor expansion T_a at a point $a \in \mathbb{R}^n$ is an arbitrary series:

$$\mathrm{T}_a = \sum_I \frac{\lambda_{I,a}}{I!} \, (x-a)^I \qquad \lambda_{I,a} \in \mathbb{R} \, .$$

A family of Taylor expansions $\{\mathrm{T}_a\}_{a \in K}$ on a set $K \subset \mathbb{R}^n$ defines functions:

$$\lambda_I : K \to \mathbb{R} \quad , \quad a \mapsto \lambda_{I,a} \, .$$

Let $K \subset \mathbb{R}^n$ be a compact set and consider a family of Taylor expansions $\{\mathrm{T}_a\}_{a \in K}$ on the points of K. This Section deals with the question of whether there exists a smooth function $f \in \mathscr{C}^\infty(\mathbb{R}^n)$ such that $\mathrm{T}_a f = \mathrm{T}_a$, for all $a \in K$.

A necessary condition is given by Taylor's Theorem: if f is a smooth function on $\mathbb{R}^n$ and $K \subset \mathbb{R}^n$ is a compact set, then for any $m \in \mathbb{N}$ and $\epsilon > 0$, there exists $\delta > 0$ such that:

$$x, y \in K \, , \; \|x-y\| < \delta \quad \Rightarrow \quad \left| f(y) - \sum_{|J|=0}^{m} \frac{D_J f(x)}{J!} (y-x)^J \right| \leq \epsilon \, \|y-x\|^m \, . \tag{1.1.0.1}$$

Whitney's Theorem provides a sufficient condition. Here, we state the result (its proof can be found in [50] or [54]) and derive some corollaries that are used later on.

Whitney's Extension Theorem: *Let $K \subset \mathbb{R}^n$ be a compact set and let $\{\mathrm{T}_a\}_{a \in K}$ be a family of Taylor expansions on K.*

There exists $f \in \mathscr{C}^\infty(\mathbb{R}^n)$ such that $\mathrm{T}_a f = \mathrm{T}_a$ for any $a \in K$

if and only if the following condition (that we will refer to as "Taylor's condition"*) holds:*

For any given I, $m \in \mathbb{N}$, and $\epsilon > 0$, there exists $\delta > 0$ such that:

$$x,y \in K \ , \ \|x-y\| < \delta \quad \Rightarrow \quad \left| \lambda_{I,y} - \sum_{|J|=0}^{m} \frac{\lambda_{I+J,x}}{J!}\,(y-x)^J \right| \le \epsilon\,\|y-x\|^m \ .$$

Corollary 1.1.1 (Borel's theorem)**.** *Given a Taylor expansion* $\mathtt{T}_a$ *at a point* $a \in \mathbb{R}^n$*, there always exists* $f \in \mathscr{C}^\infty(\mathbb{R}^n)$ *such that:*

$$\mathtt{T}_a f = \mathtt{T}_a \ .$$

Proposition 1.1.2. *Let* $a_k \to 0$ *be a sequence of points converging to the origin of* $\mathbb{R}^n$*.*

Assume there exist $c \in \mathbb{N}$ *such that:*

$$\|a_k\|, \|a_l\| < c\|a_k - a_l\| \qquad \forall\, k \neq l \ ,$$

and let $\{\mathtt{T}_{a_k}, \mathtt{T}_0 = 0\}_{k\in\mathbb{N}}$ *be a continuous family of Taylor expansions on* (a_k)*.*

If Taylor's condition holds at the origin; i.e., for any I*,* m*,* ϵ*, there exists* $\nu \in \mathbb{N}$ *such that :*

$$k > \nu \quad \Rightarrow \quad |\lambda_{I,a_k}| \le \epsilon\,\|a_k\|^m$$

then there exists a global smooth function $f \in \mathscr{C}^\infty(\mathbb{R}^n)$ *realizing those expansions:*

$$\mathtt{T}_{a_k} f = \mathtt{T}_{a_k} \qquad , \qquad \mathtt{T}_0 f = 0 \ .$$

Proof: We can assume $|(a_k - a_l)^J| < 1$, for any multi-index J and any k,l.

Given I, m, ϵ, the hypothesis allows to find $\nu \in \mathbb{N}$ such that, for any J with $|J| \le m$:

$$k > \nu \quad \Rightarrow \quad |\lambda_{I+J,a_k}| \le \epsilon\,\|a_k\|^m \ .$$

Hence, for any $k,l > \nu$:

$$\begin{aligned}
\left| \lambda_{I,a_k} - \sum_{|J|=0}^{m} \lambda_{I+J,a_l} \frac{(a_k-a_l)^J}{J!} \right| &\le |\lambda_{I,a_k}| + \sum_{|J|=0}^{m} |\lambda_{I+J,a_l}| \frac{|(a_k-a_l)^J|}{J!} \\
&\le \epsilon\,\|a_k\|^m + \epsilon\,\|a_l\|^m \sum_{|J|=0}^{m} \frac{1}{J!} \\
&\le \epsilon \left(c^m \|a_k - a_l\|^m + c^m M \|a_k - a_l\|^m \right) \le \bar{\epsilon}\,\|a_k - a_l\|^m \ .
\end{aligned}$$

Therefore, Whitney's Theorem applies and the thesis follows. $\square$

Proposition 1.1.3. *Let $K \subset \mathbb{R}^n$ be the truncated cone on $\mathbb{R}^n$ defined by the equations:*

$$K := \{x \in \mathbb{R}^n : \ x_1^2 + \ldots + x_{n-1}^2 \le x_n^2 \quad , \quad |x_n| \le 1\ \} \tag{1.1.0.2}$$

and let $K_1 := K \cap \{x_n \ge 0\}$ and $K_2 := K \cap \{x_n \le 0\}$.

If $u, v \in \mathscr{C}^\infty(\mathbb{R}^n)$ are smooth functions with the same Taylor expansion at the origin, $\mathtt{T}_0 u = \mathtt{T}_0 v$, then there exists $f \in \mathscr{C}^\infty(\mathbb{R}^n)$ such that

$$f_{|K_1} = u \quad , \quad f_{|K_2} = v \ .$$

Proof: It is enough to argue the case $v = 0$. To do so, let us check we can apply Whitney's Theorem to the following family of Taylor expansions:

$$\mathtt{T}_a := \begin{cases} \mathtt{T}_a u \ , & \text{if } a \in K_1 \\ \ \ 0 \ , & \text{if } a \in K_2 \end{cases} \quad .$$

Taylor's condition trivially holds whenever $x, y \in K_1$ or $x, y \in K_2$. If $x \in K_1$ and $y \in K_2$ (the case where x and y are interchanged is analogous), then:

$$\|x\|, \|y\| \le \|y - x\| \ .$$

Given I, m, ϵ, the hypothesis $\mathtt{T}_0 u = 0$ implies there exists $\delta > 0$ such that, for any J with $|J| \le m$,

$$x \in K_1 \ , \ \|x\| \le \delta \quad \Rightarrow \quad |\lambda_{I+J,x}| \le \epsilon \, \|x\|^m \ .$$

A smaller δ, if necessary, also guarantees $|(y-x)^J| < 1$ whenever $\|x - y\| < \delta$ and $|J| \le m$.

Therefore, whenever $x \in K_1$ and $y \in K_2$ satisfy $\|x - y\| \le \delta$,

$$\left| \lambda_{I,y} - \sum_{|J| \le m} \frac{\lambda_{I+J,x}}{J!} (y-x)^J \right| \le 0 + \sum_{|J| \le m} \big| \lambda_{I+J,x} \big| \ \frac{|(y-x)^J|}{J!} \ \le \ \epsilon \, \|x\|^m \left(\sum_{|J|=0}^{m} \frac{1}{J!} \right) \ \le \ \bar{\epsilon} \|y - x\|^m .$$

$\square$

1.2 Differential operators and spaces of jets

In this Section we define the notion of differential operator (of infinite order) and the spaces of ∞-jets. We assume a basic knowledge of the bundles of jets of finite order.

Definition. A smooth map $\pi\colon F \to X$ will be called a fibre bundle if it is a submersion.

If $\overline{\pi}\colon \overline{F} \to X$ is another fibre bundle, a morphism of bundles $P\colon F \to \overline{F}$ is a smooth map such that $\overline{\pi} \circ P = \pi$.

Let $F, \overline{F} \to X$ be fibre bundles over a smooth manifold X.

Definition. A differential operator $P : F \rightsquigarrow \overline{F}$ of finite order $\leq k$ is a morphism of bundles:

$$P : J^k F \to \overline{F}$$

where $J^k F \to X$ denotes the bundle of k-jets of sections of F.

In order to deal with differential operators of arbitrary order, it will be necessary to introduce the space of ∞-jets of sections, $J^\infty F$. As the category of smooth manifolds does not possess inverse limits, the smooth structure of this space will be worked out in the larger category of ringed spaces:

Definition. A ringed space is a pair $(X, \mathscr{O}_X)$, where X is a topological space and $\mathscr{O}_X$ is a subalgebra of the sheaf of real-valued continuous functions on X.

A morphism of ringed spaces $\varphi\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a continuous map $\varphi\colon X \to Y$ such that composition with φ induces a morphism of sheaves:

$$\varphi^* : \mathscr{O}_Y \to \varphi_* \mathscr{O}_X \ ,$$

that is, for any open set $V \subset Y$ and any function $f \in \mathscr{O}_Y(V)$, the composition $f \circ \varphi$ lies in $\mathscr{O}_X(\varphi^{-1}V)$.

Example. Any smooth manifold X is a ringed space, where $\mathscr{O}_X = \mathscr{C}^\infty_X$ is the sheaf of smooth real-valued functions. If X and Y are smooth manifolds, a morphism of ringed spaces $X \to Y$ is just a smooth map.

In the category of ringed spaces there exist inverse limits; as a particular case, consider a sequence of smooth manifolds and smooth maps between them:

$$\ldots \to X_{k+1} \xrightarrow{\varphi_{k+1}} X_k \xrightarrow{\varphi_k} X_{k-1} \to \ldots \ .$$

Definition. Its inverse limit is the ringed space $(X_\infty, \mathcal{O}_\infty)$ defined as follows:

- The underlying topological space is the inverse limit of the topological spaces X_k; i.e., the set:

$$X_\infty := \varprojlim X_k$$

endowed with the minimum topology for which the canonical projections $\pi_k \colon X_\infty \to X_k$ are continuous.

- On any open set $U \subset X_\infty$, the "smooth" functions $\mathcal{O}_\infty(U)$ are those continuous maps $f \colon U \to \mathbb{R}$ that locally factor through a smooth function defined on some X_k.

That is, a continuous map $f \colon U \to \mathbb{R}$ lies in $\mathcal{O}_\infty(U)$ if and only if for any point $x \in U$, there exist an open neighbourhood $x \in V$ and a smooth map $f_k \colon X_k \to \mathbb{R}$ such that the following triangle commutes:

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr, "\pi_k"'] & & \mathbb{R} \\ & X_k \arrow[ur, "f_k"'] & \end{tikzcd} .$$

A morphism of ringed spaces $Y \to X_\infty$ or $X_\infty \to Z$, where Y, Z are smooth manifolds, is called a smooth map.

Example. The space $J^\infty F$ of ∞-jets of sections of a fibre bundle F is the inverse limit of the sequence of k-jets prolongations:

$$\ldots \to J^k F \to J^{k-1} F \to \ldots \to F \to X .$$

If $\{N_i\}_{i\in\mathbb{N}}$ is a countable family of finite dimensional $\mathbb{R}$-vector spaces, the vector space $\prod_{i=1}^\infty N_i$ is the inverse limit of the projections:

$$\ldots \to \prod_{i=1}^{k+1} N_i \to \prod_{i=1}^{k} N_i \to \ldots \to N_2 \times N_1 \to N_1 .$$

Universal Property. *For any smooth manifold Y, the projections $\pi_k \colon X_\infty \to X_k$ induce a bijection:*

$$\mathrm{Hom}(Y, X_\infty) \ = \ \varprojlim \ \mathrm{Hom}(Y, X_k) ,$$

where $\mathrm{Hom}(\_,\_)$ *denotes the set of smooth maps.*

Proof: The projections π_k are smooth maps, so one inclusion follows. As for the other, let $\varphi \colon Y \to X_\infty$ be a continuous map such that $\pi_k \circ \varphi$ is smooth, for any $k \in \mathbb{N}$.

Let $f \in \mathscr{O}_\infty(U)$ be a smooth function and let $y \in \varphi^{-1}(U)$. On a neighbourhood V of $\varphi(y)$, there exists $f_k : X_k \to \mathbb{R}$ such that $f = f_k \circ \pi_k$, and therefore:

$$\varphi^* f = \varphi^*(f_k \circ \pi_k) = (\pi_k \circ \varphi)^* f_k$$

that is smooth because $\pi_k \circ \varphi$ is a smooth map. □

Proposition 1.2.1. *Let Z be a smooth manifold. A continuous map $\varphi\colon X_\infty \to Z$ is smooth if and only if it locally factors through a smooth map defined on some X_k.*

Proof: Let $\varphi\colon X_\infty \to Z$ be a smooth map; let $x \in X_\infty$ be a point and let $(U, z_1, \dots, z_n)$ be a coordinate chart around $\varphi(x)$ in Z. Each of the functions $z_1 \circ \varphi, \dots, z_n \circ \varphi \in \mathscr{O}_\infty(\varphi^{-1}U)$ locally factors through some X_j; as they are a finite number, there exists $k \in \mathbb{N}$ and an open neighbourhood V of x such that all of them, when restricted to V, factor through X_k. Hence, $\varphi_{|V} = (\varphi_k \circ \pi_k)_{|V}$, where $\varphi_k = (z_1 \circ \varphi, \dots, z_n \circ \varphi)$.

Conversely, let f be a smooth function on $U \subset Z$. On a neighbourhood of any point in $\varphi^{-1}U$, it holds $\varphi = \varphi_k \circ \pi_k$, so that $\varphi^* f = f \circ \varphi = f \circ \varphi_k \circ \pi_k = (\varphi_k^* f) \circ \pi_k$ on that neighbourhood. Hence, $\varphi^* f$ is a smooth function on X_∞ and φ is a smooth map. □

Definition. A differential operator $P : F \rightsquigarrow \overline{F}$ is a morphism of ringed spaces over X:

$$P : J^\infty F \to \overline{F} \ .$$

Due to the previous Proposition, differential operators have finite order locally on $J^\infty F$.

1.3 Slovák's characterization of differential operators

Let T be a smooth manifold and let $X_T := T \times X$.

Any open set $U \subset X_T$ can be thought as a family of open sets $U_t \subset X$, where U_t is the fibre of $U \to T$ over $t \in T$.

A family of sections $\{s_t : U_t \to F\}_{t \in T}$ defines a map:

$$s : U \to F \quad , \quad s(t,x) := s_t(x)$$

and $\{s_t\}_{t \in T}$ is said smooth (with respect to the parameters $t \in T$) precisely when the map $s : U \to F$ is smooth:

Definition. A smooth family of sections of F parametrized by T is a section of F with support on an open set U of X_T:

$$\begin{tikzcd} & F \arrow[d] \\ X_T \supset U \arrow[ur, "s"] \arrow[r] & X \end{tikzcd}$$

Example. Let $X = \mathbb{R}^n$, $F = \mathbb{R}^r \times \mathbb{R}^n$ and let $T := Pol_k(\mathbb{R}^n, \mathbb{R}^r)$ be the smooth manifold of polynomial maps $f : \mathbb{R}^n \to \mathbb{R}^r$ of degree less or equal than k.

The universal family ξ is defined in $U = J^k F = Pol_k(\mathbb{R}^n, \mathbb{R}^r) \times \mathbb{R}^n$ by the formula:

$$\xi : U \to F = \mathbb{R}^r \times \mathbb{R}^n \quad , \quad (f, x) \mapsto (f(x), x) \, ;$$

that is, $\xi_f : \mathbb{R}^n = X \to F = \mathbb{R}^r \times \mathbb{R}^n$ is the section defined by the polynomial f.

Definition. A morphism of sheaves $\phi : \mathscr{F} \to \overline{\mathscr{F}}$ is regular if, for any smooth family of sections $\{s_t : U_t \to F\}_{t \in T}$, the family $\{\phi(s_t) : U_t \to \overline{F}\}_{t \in T}$ is also smooth.

Example. Any differential operator $P : F \rightsquigarrow \overline{F}$ allows to define a morphism of sheaves:

$$\phi_P : \mathscr{F} \to \overline{\mathscr{F}} \qquad , \qquad \phi_P(s)(x) := P(j^\infty_x s)$$

that is regular.

Arbitrary morphisms of sheaves

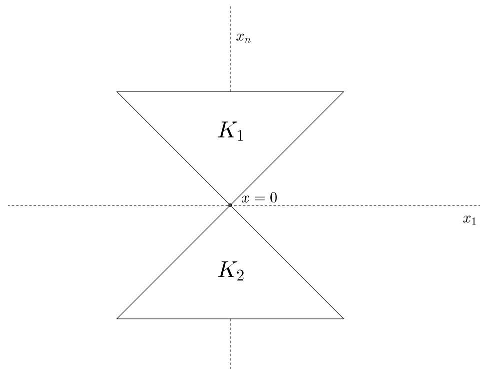
Let $F, \overline{F} \to X$ be fibre bundles and let $\mathscr{F}$, $\overline{\mathscr{F}}$ be their sheaves of smooth sections.

Proposition 1.3.1. *Let $\phi : \mathscr{F} \to \overline{\mathscr{F}}$ be a morphism of sheaves. For any sections s, s' of $\mathscr{F}$ defined on a neighbourhood of a point $x \in X$:*

$$j^\infty_x s = j^\infty_x s' \qquad \Rightarrow \qquad \phi(s)(x) = \phi(s')(x) \, .$$

Proof: As the statement is local, we can suppose $x = 0$ is the origin of $X = \mathbb{R}^n$ and $F = \mathbb{R}^r \times \mathbb{R}^n$ is trivial. We can also assume $\overline{F} = \mathbb{R} \times \mathbb{R}^n$ is trivial, with one-dimensional fibres.

Hence, let $s \equiv (s_1, \dots, s_r), s' \equiv (s'_1, \dots, s'_r) : \mathbb{R}^n \to \mathbb{R}^r$ be smooth maps with the same ∞-jet at the origin. Let $K = K_1 \cup K_2 \subset \mathbb{R}^n$ be a truncated cone as in (1.1.0.2).



In this situation, Lemma 1.1.3 proves the existence of smooth functions $f_1, \dots f_r$ on $\mathbb{R}^n$ such that:

$$f_{i|K_1} = s_{i|K_1} \quad , \quad f_{i|K_2} = s'_{i|K_2} \qquad i = 1, \dots m \ .$$

The section $f \equiv (f_1, \dots, f_r)$ satisfies $f_{|K_1} = s_{|K_1}$, $f_{|K_2} = s'_{|K_2}$, and, consequently, as ϕ commutes with restrictions to open sets:

$$\phi(f)_{|\overset{\circ}{K}_1} = \phi(s)_{|\overset{\circ}{K}_1} \qquad , \qquad \phi(f)_{|\overset{\circ}{K}_2} = \phi(s')_{|\overset{\circ}{K}_2} \ .$$

By continuity,

$$\phi(s)(x) = \phi(f)(x) = \phi(s')(x) \ .$$

$\square$

As a consequence, if $\phi \colon \mathscr{F} \to \overline{\mathscr{F}}$ is a morphism of sheaves, the following map (between sets) is well-defined:

$$P_\phi \colon J^\infty F \to \overline{F} \quad , \quad P_\phi(j^\infty_x s) := \phi(s)(x) \ .$$

Proposition 1.3.2. *Let $\phi \colon \mathscr{F} \to \overline{\mathscr{F}}$ be a morphism of sheaves. If s, s' are sections of $\mathscr{F}$ and $x_k \to x$ is a sequence converging to a point $x \in X$, then,*

$$j^k_{x_k} s = j^k_{x_k} s' \ , \ \forall \ k \in \mathbb{N} \quad \Rightarrow \quad \exists \ k_0 \ : \ \phi(s)(x_k) = \phi(s')(x_k) \quad \forall \ k > k_0 \ .$$

Proof: Again, we can suppose $X = \mathbb{R}^n$ and $F = \mathbb{R}^r \times \mathbb{R}^n$, $\overline{F} = \mathbb{R} \times \mathbb{R}^n$ are trivial bundles.

If the statement is not true, taking a subsequence we can assume there exists s, s' such that, for any k:

$$j^k_{x_k} s = j^k_{x_k} s' \quad , \quad \phi(s)(x_k) \neq \phi(s')(x_k) \ .$$

Let us consider another sequence $y_k \to x$ such that:

$$|\phi(s)(x_k) - \phi(s')(y_k)| > k \, \|y_k - x_k\| \ . \tag{1.3.0.3}$$

$$x_k \neq y_l \qquad \forall \, k,l \ . \tag{1.3.0.4}$$

Let us now apply Whitney's Extension Theorem on the compact $(x_k) \cup (y_k) \cup \{0\}$ to the family of jets:

$$\{ \, j^\infty_{x_k} s \ , \ j^\infty_{y_k} s' \ , \ j^\infty_x s = j^\infty_x s' \, \} \ . \tag{1.3.0.5}$$

Due to Taylor's Theorem (1.1.0.1), given I, m, ϵ, there exists $\delta > 0$ such that, for any $i = 1, \ldots, m$ and any points a, b in the compact, the condition $\|a - b\| < \delta$ implies:

$$\left| (D_I s_i)(a) - \sum_{|J|=0}^{m} (D_{I+J} s_i)(b) \frac{(a-b)^J}{J!} \right| \leq \epsilon \, \|a - b\|^m \tag{1.3.0.6}$$

$$\left| (D_I s'_i)(a) - \sum_{|J|=0}^{m} (D_{I+J} s'_i)(b) \frac{(a-b)^J}{J!} \right| \leq \epsilon \, \|a - b\|^m \tag{1.3.0.7}$$

A smaller δ, if necessary, also guarantees:

$$\|x_k - y_l\| < \delta \quad \Rightarrow \quad k, l > |I| + m =: M \ ,$$

and this allows to invoke Whitney's Theorem:

- If $\|x_k - x_l\|, \|y_k - y_l\| < \delta$, then (1.3.0.7) is just Taylor's condition.
- If $\|x_k - y_l\| < \delta$, then $k, l > M = |I| + m$; as $j^M_{x_k} s = j^M_{x_k} s'$, it holds:

$$\left| (D_I s')(y_l) - \sum_{|J|=0}^{m} (D_{I+J} s)(x_k) \frac{(y_k - x_k)^J}{J!} \right| = \left| (D_I s')(y_l) - \sum_{|J|=0}^{m} (D_{I+J} s')(x_k) \frac{(y_k - x_k)^J}{J!} \right|$$

  and inequality (1.3.0.7) proves this quantity is less or equal than $\epsilon \, \|y_l - x_k\|^m$.

Therefore, there exists a smooth map $f : \mathbb{R}^n \to \mathbb{R}^r$ realizing the family of jets (1.3.0.5). Due to Proposition 1.3.1, this map f satisfies:

$$\phi(f)(x_k) = \phi(s)(x_k) \quad , \quad \phi(f)(y_k) = \phi(s')(y_k)$$

and (1.3.0.3) contradicts the smoothness of $\phi(f)$. □

Regular morphisms of sheaves

Let $\phi\colon \mathscr{F} \to \overline{\mathscr{F}}$ be a regular morphism of sheaves. By Proposition 1.3.1, the following map between sets is well-defined:

$$P_\phi\colon J^\infty F \to \overline{F} \quad , \quad P_\phi(j^\infty_x s) := \phi(s)(x) \ .$$

Lemma 1.3.3. *If $\phi\colon \mathscr{F} \to \overline{\mathscr{F}}$ is a regular morphism of sheaves, then the map $P_\phi\colon J^\infty F \to \overline{F}$ locally factors through some finite jet space.*

That is, for any $j^\infty_x s \in J^\infty F$ there exist an open neighbourhood V, a natural number $k \in \mathbb{N}$ and a commutative triangle of maps:

$$\begin{tikzcd} V \arrow[rr, "P_\phi"] \arrow[dr, "\pi_k"'] & & \overline{F} \\ & J^k F \arrow[ur] & \end{tikzcd} \ .$$

Proof: It is a local problem, so we can assume $X = \mathbb{R}^n$, $F = \mathbb{R}^r \times \mathbb{R}^n$ and $x = 0$. Moreover, we can assume the section $s\colon \mathbb{R}^n \to \mathbb{R}^r$ representing the jet $j^\infty_0 s$ is the zero section $s = 0$.

For each $k \in \mathbb{N}$, consider the following neighbourhood of $j^\infty_0 s$:

$$U_k := \left[j^\infty_x f \in JF \ / \quad \|x\| \le \frac{1}{2^k} \ , \ |D_I f_i(x)| \le \left(\frac{1}{2^k}\right)^k \ , \ \forall\, |I| \le k \ , \ i = 1,\dots,m \right] \ .$$

If the thesis is not true, for any $k \in \mathbb{N}$ we can find $j^\infty_{x_k} f^k, j^\infty_{x_k} h^k \in U_k$ such that

$$j^k_{x_k} f^k = j^k_{x_k} h^k \quad \text{but} \quad \phi(f^k)(x_k) \neq \phi(h^k)(x_k) \ .$$

If we could find smooth sections f, h of F such that $j^\infty_{x_k} f = j^\infty_{x_k} f^k$, $j^\infty_{x_k} h = j^\infty_{x_k} h^k$, then the above statement contradicts Proposition 1.3.2, as it would be, for any $k \in \mathbb{N}$:

$$j^k_{x_k} f = j^k_{x_k} h \quad \text{but} \quad \phi(f)(x_k) = \phi(f^k)(x_k) \neq \phi(h^k)(x_k) = \phi(h)(x_k) \ .$$

But, in order to apply Whitney's Theorem and extend $j^\infty_{x_k} f^k, j^\infty_{x_k} h^k$, the points x_k may be inconveniently placed. To overcome this difficulty, let us parametrize by $\mathbb{R}$ and consider the points

$$z_k := \left(\frac{1}{2^k} \ , \ x_k\right) \in \mathbb{R} \times \mathbb{R}^n \ ,$$

that satisfy:

$$\frac{1}{2^k} \le \|z_k\| \le \frac{1}{2^{k-1}} \quad , \quad \|z_k\|, \|z_l\| < 4\|z_k - z_l\| \qquad \forall\, k \neq l.$$

Let us extend each section s of F over $X = \mathbb{R}^n$ to the constant family $s(t,x) := s(x)$ over $\mathbb{R} \times X = \mathbb{R} \times \mathbb{R}^n$.

For any fixed I, m, a sufficiently large k assures:

$$|D_I f_i^k(z_k)| = |D_I f_i^k(x_k)| < \left(\frac{1}{2^k}\right)^k \le \left(\frac{1}{2^k}\right)^{k-m} \|z_k\|^m = \epsilon\, \|z_k\|^m$$

and idem for the h_i^k.

In this situation, Proposition 1.1.2 proves there exist smooth sections $f = (f_1, \ldots, f_r)$, $h = (h_1, \ldots, h_r)$ such that, for any k:

$$j^\infty_{z_k} f = j^\infty_{z_k} f^k \quad , \quad j^\infty_{z_k} h = j^\infty_{z_k} h^k \quad , \quad \forall\, k \in \mathbb{N}\,.$$

Thus, these sections f, h of F satisfy:

$$j^k_{z_k} f = j^k_{z_k} h \quad \text{but} \quad \phi(f)(z_k) \neq \phi(h)(z_k) \quad , \quad \forall\, k \in \mathbb{N}$$

in contradiction with Proposition 1.3.2. □

The following statement is inspired in more general results due to Slovak ([44]):

Theorem 1.3.4. *Let $\mathscr{F}, \overline{\mathscr{F}}$ be the sheaves of smooth sections of two fibre bundles $F, \overline{F}$ over a smooth manifold X.*

The map $P \mapsto \phi_P$ establishes a bijection:

$$\mathrm{Hom}_{reg}(\mathscr{F}, \overline{\mathscr{F}}) = \mathrm{Diff}(F, \overline{F})$$

where $\mathrm{Hom}_{reg}(\mathscr{F}, \overline{\mathscr{F}})$ and $\mathrm{Diff}(F, \overline{F})$ stand for the sets of regular morphisms of sheaves and differential operators, respectively.

Proof: Let $\phi \colon \mathscr{F} \to \overline{\mathscr{F}}$ be a regular morphism of sheaves.

In virtue of Lemma 1.3.3, any point in $J^\infty F$ has an open neighbourhood V such that the following triangle commutes:

$$\begin{tikzcd} V \arrow[rr, "P_\phi"] \arrow[dr, "\pi_k"'] & & \overline{F} \\ & J^k F \arrow[ur, "P_k"'] & \end{tikzcd} .$$

It only rests to check that $P_k : J^kF \to \overline{F}$, $j^k_x s \mapsto \phi(s)(x)$ is smooth, and, to this end, we can suppose $X = \mathbb{R}^n$ and $F = \mathbb{R}^r \times \mathbb{R}^n$ is trivial.

Let ξ be the universal family of k-jets (Example 1.3). The smoothness of P_k follows from the regularity of ϕ, because:

$$P_k(j^k_x s) = \phi(\xi_f)(x) ,$$

where $f : X \to F$ is the only polynomial whose k-jet at x is $j^k_x s$. $\square$

Chapter 2

Natural bundles

In this thesis, we consider tensors $T(g)$ intrinsically associated to a pseudo-Riemannian metric g; that is, tensors such that the assignment $g \mapsto T(g)$ is independent of coordinates.

The concept of natural morphism formalizes the idea of a morphism of bundles whose definition does not depend on choices of coordinates. As a change of coordinates is, essentially, the same thing as a diffeomorphism, natural morphisms are defined as morphisms of bundles equivariant with respect to the action of diffeomorphisms of the base manifold.

The theory of natural bundles is classical, and its modern development is mainly due o the work of Nijenhuis ([39], [40]), Atiyah-Bott-Patodi ([6]), Palais-Terng ([41], [48]), Epstein-Thurston ([14]) and Slovák ([45]), among many others. Our presentation differs from the standard references (the book by Kolář-Michor-Slovák, [24]) but is equivalent to it.

2.1 Notion of natural bundle

Let us fix a smooth manifold X of dimension n.

Let $\mathrm{Diff}X$ denote the set of diffeomorphisms $\tau\colon U \to V$ between open sets in X. If $\pi\colon F \to X$ is a fibre bundle, a lift of a diffeomorphism $\tau\colon U \to V$ between open sets on X is any diffeomorphism $\tau_*\colon F_U := \pi^{-1}(U) \to F_V := \pi^{-1}(U)$ such that the following square is commutative:

$$\begin{array}{ccc} F_U & \xrightarrow{\tau_*} & F_V \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ U & \xrightarrow{\tau} & V \end{array} .$$

Definition. A natural bundle over X is a fibre bundle $F \to X$ together with a lifting of diffeomorphisms:

$$\mathrm{Diff}X \xrightarrow{*} \mathrm{Diff}F \quad , \quad \tau \mapsto \tau_*$$

that satisfies:

- *Functoriality:* $\mathrm{Id}_* = \mathrm{Id}$ and $(\tau \circ \tau')_* = \tau_* \circ \tau'_*$.
- *Locality:* For any diffeomorphism $\tau \colon U \to V$ between open sets on X, and for any open set $U' \subset U$:
$$(\tau_{|U'})_* = (\tau_*)_{|F_{U'}} \ .$$
- *Regularity:* If $\{\tau_t \colon U_t \to V_t\}_{t\in T}$ is a smooth family of diffeomorphisms between open sets on X, parametrized by a smooth manifold T, then $\{\tau_{t*} \colon F_{U_t} \to F_{V_t}\}_{t\in T}$ is a smooth family of diffeomorphisms between open sets on F.

Let $F, \overline{F} \to X$ be natural bundles over X. A morphism of natural bundles (or natural morphism) $\varphi \colon F \to \overline{F}$ is a morphism of bundles that commutes with the action of $\mathrm{Diff}X$; that is, such that for any diffeomorphism $\tau \colon U \to V$ between open sets on X, the following square is commutative:

$$\begin{tikzcd} F_U \arrow[r, "\varphi"] \arrow[d, "\tau_*"'] & \overline{F}_U \arrow[d, "\tau_*"] \\ F_V \arrow[r, "\varphi"] & \overline{F}_V \end{tikzcd} \ .$$

The regularity condition is introduced in the definition for simplicity, but it can be deduced from the other two ([14]).

Examples. The trivial bundle $F = Y \times X \to X$, with the lifting $\tau_*(y,x) = (y, \tau(x))$, is a natural bundle.

The tangent bundle $TX \to X$ is a natural bundle: the lifting of a diffeomorphism $\tau \colon U \to V$ is its tangent linear map $\tau_* \colon TU \to TV$. More generally, any bundle of tensors $\otimes^p T^*X \otimes^q TX$ is natural.

If $F \to X$ is a natural bundle, then $J^kF \to X$ is also a natural bundle, for any $k \in \mathbb{N}$.

Definition. A natural bundle $F \to X$ has order $\leq k$ if for any diffeomorphims $\tau, \tau' \colon U \to V$ defined between open sets of X, and any $x \in U$, it holds

$$j^k_x \tau = j^k_x \tau' \quad \Rightarrow \quad \tau_* = \tau'_* \quad \text{when restricted to the fibre } F_x$$

Although it will not be used in this memory, it can be proved that any natural bundle has finite order. To be precise, if the fibres of a natural bundle have dimension $\leq d$, then the bundle has order $\leq 2d+1$ ([14]).

Example. Any bundle of tensors is a natural bundle of order 1.

An alternative definition of natural bundle

The definition of natural bundle presented so far was indicated to us by Juan B. Sancho Guimerá. In this Section, we briefly comment the standard definition in the literature, originally formulated by Nijenhuis ([40]), and the equivalence between them.

Let $\mathbf{Man}_n$ be the category of smooth manifolds of dimension n and local diffeomorphisms between them.

On the other hand, let **Bund** be the category whose objects are fibre bundles $F \to X$, and whose morphisms $f:\{F\to X\} \longrightarrow \{\overline{F}\to\overline{X}\}$ are smooth maps $f: F\to\overline{F}$ transforming each fibre of F into a fibre of $\overline{F}$, so that that there exists a smooth map $\overline{f}: X\to\overline{X}$ making the square commutative:

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, "\pi"'] & \overline{F} \arrow[d, "\overline{\pi}"] \\ X \arrow[r, "\overline{f}"] & \overline{X} \end{tikzcd} .$$

Definition. A natural bundle is a covariant functor:

$$\mathfrak{F}\colon \mathbf{Man}_n \to \mathbf{Bund}$$

satisfying the following properties:

- If $\mathfrak{B}\colon \mathbf{Bund}\to\mathbf{Man}$ denotes the base functor, $\mathfrak{B}(F\to X) := X$, then the composition $\mathfrak{B}\circ\mathfrak{F}\colon \mathbf{Man}_n\to\mathbf{Man}_n$ is the identity functor $\mathrm{Id}_{\mathbf{Man}_n}$.

  For any smooth n-manifold X, let us denote $\mathfrak{F}(X) = \{\pi_X : \mathfrak{F}_X \to X\}$.

- *Locality:* For any open inclusion $i\colon U\hookrightarrow X$, the bundle $\mathfrak{F}_U$ is identified with $\pi_X^{-1}(U)$ via the map $\mathfrak{F}(i)$.

- *Regularity:* If $\{f_t\colon X_t\to Y_t\}_{t\in T}$ is a smooth family of local diffeomorphisms between n-manifolds, parametrized by a smooth manifold T, then $\{\mathfrak{F}(f_t)\colon \mathfrak{F}(X_t)\to\mathfrak{F}(Y_t)\}_{t\in T}$ is also a smooth family.

A morphism of natural bundles is a morphism of functors $\Phi\colon \mathfrak{F}\to\overline{\mathfrak{F}}$.

Definition. A natural bundle $\mathscr{F}$ has order $\leq k$ if for any local diffeomorphisms between n-manifolds $f, g\colon X \to Y$ and any point $x \in X$, the following holds:

$$j^k_x f = j^k_x g \quad \Rightarrow \quad \mathfrak{F}(f) = \mathfrak{F}(g) \ \text{ when restricted to the fibre } (\mathfrak{F}_X)_x\ .$$

The proof of the following Proposition is not difficult (see Remark in page 25):

Proposition 2.1.1. *Let X be a smooth manifold of dimension n, and let* $\mathbf{Nat}_X$ *and* $\mathbf{Nat}_n$ *be the categories of natural bundles defined in page 15 and page 17, respectively.*

The functor $\mathfrak{F} \mapsto \mathfrak{F}(X)$ establishes an equivalence of categories:

$$\mathbf{Nat}_n \;=\!=\!=\; \mathbf{Nat}_X$$

In this memory, we will use the first definition, due to its simplicity and its emphasis in the property of lifting of diffeomorphisms. The advantage of the second one lies in the fact that it makes clear that each natural bundle is simultaneously defined over all n-manifolds.

2.2 Natural morphisms

Let $x \in X$ be a point and let Diff_x denote the group of germs of diffeomorphisms of X leaving the point x fixed.

If $F \to X$ is a natural bundle, the group Diff_x acts on the fibre F_x, due to the natural lift. If $\varphi\colon F \to \overline{F}$ is a morphism of natural bundles then its restriction to the fibre, $\varphi_x\colon F_x \to \overline{F}_x$, is a Diff_x-equivariant smooth map.

Proposition 2.2.1. *Let $F, \overline{F} \to X$ be natural bundles. Given any point $x \in X$, the assignment $\varphi \mapsto \varphi_x$ defines a bijection:*

$$\left\{ \begin{array}{c} \text{Morphisms of natural bundles} \\ \\ F \to \overline{F} \end{array} \right\} \;=\!=\!=\; \left\{ \begin{array}{c} \mathrm{Diff}_x - \text{equivariant smooth maps} \\ \\ F_x \to \overline{F}_x \end{array} \right\} .$$

Proof: Firstly, let us check that any natural morphism $\varphi\colon F \to \overline{F}$ is determined by its restriction $\varphi_x\colon F_x \to \overline{F}_x$ to the fibre over x.

At any other point $y \in X$, consider a diffeomorphism $\tau\colon X \to X$ such that $\tau(x) = y$; the condition $\tau_* \circ \varphi = \varphi \circ \tau_*$ restricted to F_x produces the commutative square:

$$\begin{tikzcd} F_x \arrow[r, "\varphi_x"] \arrow[d, "\tau_*"'] & \overline{F}_x \arrow[d, "\tau_*"] \\ F_y \arrow[r, "\varphi_y"] & \overline{F}_y \end{tikzcd}$$

and therefore $\varphi_y = \tau_* \circ \varphi_x \circ \tau_*^{-1}$ is determined by φ_x.

Let us now check that any Diff_x-equivariant smooth map $\phi\colon F_x \to \overline{F}_x$ is the restriction to F_x of a natural morphism $\varphi\colon F \to \overline{F}$.

To do so, define φ on each fibre F_y by the formula:

$$\varphi_y = \tau_* \circ \phi \circ \tau_*^{-1} \ ,$$

where $\tau: X \dashrightarrow X$ is any diffeomorphism such that $\tau(x) = y$.

This definition does not depend on this choice: if $\overline{\tau}\colon X \dashrightarrow X$ is another diffeomorphism such that $\overline{\tau}(x) = y$, then $\tau^{-1}\overline{\tau} \in \mathrm{Diff}_x$ so $\phi \circ (\tau_*^{-1}\overline{\tau}_*) = (\tau_*^{-1}\overline{\tau}_*) \circ \phi$, and $\tau_* \circ \phi \circ \tau_*^{-1} = \overline{\tau}_* \circ \phi \circ \overline{\tau}_*^{-1}$.

Finally, let us prove the smoothness of $\varphi\colon F \to \overline{F}$. If $y \in X$ is any point, let $V = \mathbb{R}^n$ be a chart centred on it and fix a diffeomorphism $\tau: X \supseteq U \to V \subseteq X$ such that $\tau(x) = y$.

Let $\tau_v : V = \mathbb{R}^n \longrightarrow \mathbb{R}^n = V$ be the translation with respect to $v \in \mathbb{R}^n$, so that $\tau_v\tau : U \to V$ satisfies

$$(\tau_v\tau)(x) = y + v \ .$$

As $\{\tau_v \circ \tau : U \to V\}_{v\in\mathbb{R}^n}$ is a smooth family of diffeomorphisms, its lifting $\{(\tau_v \circ \tau)_* : F_U \to F_V\}_{v\in\mathbb{R}^n}$ is also a smooth family. Then, the map under consideration:

$$\varphi\colon F_V \to \overline{F}_V \quad , \quad \varphi(s_{y+v}) = \left[(\tau_v\tau)_* \circ \phi \circ (\tau_v\tau)_*^{-1}\right](s_{y+v})$$

is smooth, because it is the composition of three smooth maps. □

Definition. Let G_x^k be the Lie group of k-jets $j_x^k\tau$ of germs of diffeomorphisms $\tau: X \dashrightarrow X$ leaving the point x fixed.

Let $G_x^\infty = \varprojlim G_x^k$ be the group of ∞-jets $j_x^\infty\tau$ of germs of diffeomorphisms $\tau: X \dashrightarrow X$ leaving the point x fixed.

Lemma 2.2.2. *Let $F \to X$ be a natural fibre bundle and let $\tau \in \mathrm{Diff}_x$.*

$$j_x^\infty\tau = j_x^\infty \mathrm{Id} \quad \Rightarrow \quad \tau_* = \mathrm{Id} \ \textit{on } F_x \ .$$

Proof: As it is a local problem, we can assume $X = \mathbb{R}^n$ and $x = 0$. Let us consider a truncated cone $K = K_1 \cup K_2$ as in Proposition 1.1.3. By this result, there exists a smooth map $\sigma \colon \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\sigma_{|\overset{o}{K}_1} = \tau_{|\overset{o}{K}_1} \qquad \text{and} \qquad \sigma_{\overset{o}{K}_2} = \mathrm{Id}_{|\overset{o}{K}_2} \ .$$

Observe σ is a diffeomorphism in a neighbourhood of 0 because $j^\infty_0 \sigma = j^\infty_0 \mathrm{Id}$.

In a neighbourhood of the origin, for any section s of F it holds:

$$\sigma_*(s)_{|\overset{o}{K}_1} = \tau_*(s)_{|\overset{o}{K}_1} \qquad , \qquad \sigma_*(s)_{|\overset{o}{K}_2} = \mathrm{Id}_*(s)_{|\overset{o}{K}_2} = s_{|\overset{o}{K}_2}$$

so, by continuity, $\tau_*(s)(0) = s(0)$, that is, τ_* is the identity on $F_{x=0}$. □

As a consequence, on any natural bundle $F \to X$, the action of Diff_x on the fibre F_x factors through the quotient $\mathrm{Diff}_x \to G^\infty_x$, $\tau \mapsto j^\infty_x \tau$.

Thus, we can reformulate Proposition 2.2.1 as follows:

Proposition 2.2.3. *Let $F, \overline{F} \to X$ be natural bundles. Given any point $x \in X$, the assignment $\varphi \mapsto \varphi_x$ defines a bijection:*

$$\left\{ \begin{matrix} \text{Morphisms of natural bundles} \\ \\ F \to \overline{F} \end{matrix} \right\} ═══ \left\{ \begin{matrix} G^\infty_x - \text{equivariant smooth maps} \\ \\ F_x \to \overline{F}_x \end{matrix} \right\} .$$

The condition of a natural bundle $F \to X$ being of order $\leq k$ amounts to saying that the fibre F_x over any point $x \in X$ is a G^k_x-manifold, since the action of Diff_x on F_x factors through the quotient $\mathrm{Diff}_x \to G^k_x$, $\tau \to j^k_x \tau$.

Therefore, Proposition 2.2.1 may be rephrased for natural bundles of order $\leq k$:

Proposition 2.2.4 (Terng)**.** *Let $F, \overline{F} \to X$ be natural bundles of order $\leq k$. Given any point $x \in X$, the assignment $\varphi \mapsto \varphi_x$ defines a bijection:*

$$\left\{ \begin{matrix} \text{Morphisms of natural bundles} \\ \\ F \to \overline{F} \end{matrix} \right\} ═══ \left\{ \begin{matrix} G^k_x - \text{equivariant smooth maps} \\ \\ F_x \to \overline{F}_x \end{matrix} \right\} .$$

Example. The only morphisms of vector bundles $\varphi \colon TX \to TX$ that are natural are the homotheties, $\varphi(D) = \lambda D$, because:

$$\mathrm{Hom}_{\mathrm{nat}}(TX, TX) = \mathrm{Hom}_{G^1_x}(T_xX, T_xX) = \mathrm{Hom}_{Gl(T_xX)}(T_xX, T_xX) = \mathbb{R} \ .$$

2.3 Natural operators

Let $F, \overline{F} \to X$ be natural bundles.

Definition. A differential operator $P\colon F \rightsquigarrow \overline{F}$ or order $\leq r$ is natural if the morphism of bundles $P\colon J^rF \to \overline{F}$ is natural.

If $F \to X$ is a natural bundle of order $\leq k$, then $J^rF \to X$ is a natural bundle of order $\leq k+r$. Therefore, Proposition 2.2.4 implies:

Proposition 2.3.1 (Terng). *Let $F, \overline{F} \longrightarrow X$ be natural bundles of order $\leq k$. Given any point $x \in X$, the assignment $P \mapsto P_x$ defines a bijection:*

$$\left\{ \begin{array}{c} \text{Natural differential operators} \\ \\ F \rightsquigarrow \overline{F} \quad \text{of order } \leq r \end{array} \right\} \;=\!=\!=\; \left\{ \begin{array}{c} G_x^{k+r}\text{–equivariant smooth maps} \\ \\ J_x^rF \to \overline{F}_x \end{array} \right\} .$$

Examples. In some cases, this proposition allows to completely describe all the natural differential operators of a certain kind. As an illustration, let us point out that the only $\mathbb{R}$-bilinear, natural differential operators

$$TX \times TX \rightsquigarrow TX$$

are the constant multiples of the Lie bracket ([27]).

Also, the only natural differential operators

$$\Lambda^p T^*X \rightsquigarrow \Lambda^{p+1}T^*X\ , \qquad p > 0\ ,$$

are the constant multiples of the exterior differential ([25]).

Example. Let $F \to X$ be a natural bundle of order k, and let $\mathbb{R}_X := \mathbb{R} \times X \to X$ be the trivial bundle. By definition, a *scalar differential invariant* of order $\leq r$ associated to the bundle $F \to X$, is a natural differential operator $F \rightsquigarrow \mathbb{R}_X$ or order $\leq r$.

By Proposition 2.3, it holds:

$$\left\{ \begin{array}{c} \text{Scalar invariants of order } \leq \text{r} \\ \\ \text{associated to } F \to X \end{array} \right\} \;=\!=\!=\; \mathscr{C}^\infty\left(J_x^rF, \mathbb{R}\right)^{G_x^{k+r}} \;=\!=\!=\; \mathscr{C}^\infty\left((J_x^rF)/G_x^{k+r}, \mathbb{R}\right)$$

where the last equality makes sense whenever the quotient J_x^rF/G_x^{k+r} admits an adequate smooth structure. This quotient has been studied in the case of linear frames ([16]) and that of Riemannian metrics ([21]).

On the other hand, the bundle $J^\infty F \to X$ of ∞-jets of sections of F also has a lifting of diffeomorphisms; therefore, it can also be considered as a "natural bundle", with the only proviso that $J^\infty F$ has not finite dimension.

Definition. A differential operator $P\colon J^\infty F \to \overline{F}$ is natural if it is a morphism of natural bundles; that is, if it commutes with the lifting of diffeomorphisms.

The proof of the following result is similar to that of the Proposition 2.2.1:

Proposition 2.3.2. *Let $F, \overline{F} \longrightarrow X$ be natural bundles. Given any point $x \in X$, the assignment $P \mapsto P_x$ defines a bijection:*

$$\left\{ \begin{matrix} \text{Natural diferential operators} \\ F \rightsquigarrow \overline{F} \end{matrix} \right\} = \!= \left\{ \begin{matrix} G^\infty_x\text{–equivariant smooth maps} \\ J^\infty_x F \to \overline{F}_x \end{matrix} \right\} .$$

Finally, let us remark that the notion of natural differential operator between natural bundles can be expressed in terms of their sheaves of smooth sections, $\mathscr{F}$ and $\overline{\mathscr{F}}$.

The naturalness of F implies that any diffeomorphism $\tau\colon U \to V$ between open sets of X defines a bijection:

$$\mathscr{F}(U) \xrightarrow{\ \tau_*\ } \mathscr{F}(V) \quad , \qquad \tau_*(s) = \tau_* \circ s \ .$$

Definition. A natural operator $\phi\colon \mathscr{F} \to \overline{\mathscr{F}}$ is a regular morphism of sheaves such that, for any diffeomorphism $\tau\colon U \to V$ between open sets of X, the following square is commutative:

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\ \phi\ } & \overline{\mathscr{F}}(U) \\ {\scriptstyle \tau_*}\downarrow & & \downarrow{\scriptstyle \tau_*} \\ \mathscr{F}(V) & \xrightarrow{\ \phi\ } & \overline{\mathscr{F}}(V) \end{array}$$

In virtue of Slovák's Theorem 1.3.4, the previous Proposition 2.3.2 can be reformulated as follows:

Proposition 2.3.3. *Let $F, \overline{F} \longrightarrow X$ be natural bundles. Given any point $x \in X$, the assignments $\Phi_P \mapsto P \mapsto P_x$ define bijections:*

$$\left\{ \begin{matrix} \text{Natural operators} \\ \mathscr{F} \to \overline{\mathscr{F}} \end{matrix} \right\} = \left\{ \begin{matrix} \text{Natural diferential} \\ \text{operators } \ F \rightsquigarrow \overline{F} \end{matrix} \right\} = \left\{ \begin{matrix} G^\infty_x\text{–equivariant smooth maps} \\ J^\infty_x F \to \overline{F}_x \end{matrix} \right\} .$$

2.4 Theorem of equivalence

In what follows, let X be a smooth manifold of dimension n and let us fix a point $x_0 \in X$.

We have seen that, if $F \to X$ is a natural bundle of order k, then the group $G^k_{x_0}$ acts on the fibre F_{x_0}. Moreover, morphisms between natural bundles of order k are in bijection with smooth $G^k_{x_0}$-equivariant maps between the corresponding fibres.

This fact suggests an equivalence of categories, that we proceed to detail in this Section.

The universal natural bundle of order k

Definition. Let $\mathcal{U}^k$ be the smooth manifold formed by jets $j^k_x\sigma$ of diffeomorphisms $\sigma\colon X \dashrightarrow X$ that map an arbitrary point $x \in X$ into the prefixed point x_0.

The universal bundle of order k is the projection:

$$\mathcal{U}^k \longrightarrow X \qquad , \qquad j^k_x\sigma \mapsto x \ .$$

The Lie group $G^k_{x_0}$ acts on $\mathcal{U}^k$:

$$(j^k_{x_0}g)\cdot(j^k_x\sigma) := j^k_x(g\circ\sigma)$$

and, via this action, the universal bundle becomes a principal bundle of group $G^k_{x_0}$.

The universal bundle $\mathcal{U}^k \to X$ is natural of order k: the lifting of a diffeomorphism $\tau\colon U \to V$ between open sets of X is defined as:

$$\mathcal{U}^k_{|U} \xrightarrow{\ \tau_*\ } \mathcal{U}^k_{|V} \quad , \qquad \tau_*(j^k_x\sigma) := j^k_{\tau(x)}(\sigma\circ\tau^{-1}) \ .$$

Let E_0 be a $G^k_{x_0}$-manifold; that is, E_0 is a smooth manifold together with a smooth action:

$$G^k_{x_0} \times E_0 \xrightarrow{\ \cdot\ } E_0 \ .$$

The group $G^k_{x_0}$ acts on $\mathcal{U}^k \times E_0$, via the action on each factor.

Definition. For any $G^k_{x_0}$-manifold E_0, its associated bundle is defined as:

$$E = (\mathcal{U}^k \times E_0)/G^k_{x_0} \longrightarrow X \quad , \qquad [(j^k_x\sigma, e)] \mapsto x \ .$$

The associated bundle $E \to X$ is natural of order k: the lifting of a diffeomorphism $\tau\colon U \to$

V between open sets of X is defined by the formula:

$$E_U = (\mathscr{U}^k_U \times E_0)/G^k_{x_0} \xrightarrow{\tau_*} (\mathscr{U}^k_V \times E_0)/G^k_{x_0} = E_V$$

$$\tau_*\left([j^k_x\sigma, e]\right) := [\tau_*(j^k_x\sigma), e] \, .$$

This lifting is well defined because the actions of τ_* and $G^k_{x_0}$ on $\mathscr{U}^k$ commute:

$$\begin{aligned}\tau_*\left((j^k_{x_0}g)\cdot(j^k_x\sigma)\right) &= \tau_*\left(j^k_x(g\circ\sigma)\right) = j^k_{\tau(x)}\left(g\circ\sigma\circ\tau^{-1}\right) \\ &= (j^k_{x_0}g)\cdot\left(j^k_{\tau(x)}(\sigma\circ\tau^{-1})\right) = (j^k_{x_0}g)\cdot\tau_*(j^k_x\sigma) \, .\end{aligned}$$

The theorem of equivalence

Let $\mathbf{Nat}^k_X$ denote the category of natural bundles of order $\leq k$ over X, and let $G^k_{x_0}$**-Man** denote the category of $G^k_{x_0}$-manifolds and equivariant smooth maps.

Let us consider the functor "associated bundle":

$$G^k_{x_0}\text{-}\mathbf{Man} \longrightarrow \mathbf{Nat}^k_X \quad , \quad E_0 \mapsto E = (\mathscr{U}^k \times E_0)/G^k_{x_0} \, .$$

This functor transforms a morphism of $G^k_{x_0}$-manifolds $f \colon E_0 \to E'_0$ into the following morphism of natural bundles:

$$E = (\mathscr{U}^k \times E_0)/G^k_{x_0} \xrightarrow{\mathrm{Id}\times f} (\mathscr{U}^k \times E'_0)/G^k_{x_0} = E' \quad , \quad [j^k_x\sigma, e] \mapsto [j^k_x\sigma, f(e)] \, .$$

Theorem 2.4.1. *The functor "associated bundle" establishes an equivalence of categories:*

$$G^k_{x_0}\text{-}\mathbf{Man} \;=\!=\!=\; \mathbf{Nat}^k_X \, .$$

Proof. The inverse functor is the "functor fibre":

$$\mathbf{Nat}^k_X \longrightarrow G^k_{x_0}\text{-}\mathbf{Man} \quad , \quad E \mapsto E_{x_0} \, .$$

Let us check both functors are inverse to each other: if E_0 is a $G^k_{x_0}$-manifold and $E \to X$ is the associated bundle, then we have a diffeomorphism

$$\begin{array}{ccc} E_0 & =\!=\!= & \left[(\mathscr{U}^k \times E_0)/G^k_{x_0}\right]_{x_0} = E_{x_0} \\ e & \mapsto & [(j^k_{x_0}\mathrm{Id}, e)] \end{array}$$

which is $G^k_{x_0}$-equivariant:

$$j^k_{x_0}g\cdot[j^k_{x_0}\mathrm{Id},e] = g_*[j^k_{x_0}\mathrm{Id},e] = [g_*(j^k_{x_0}\mathrm{Id}),e] = [j^k_{x_0}(\mathrm{Id}\circ g^{-1}),e]$$
$$= [j^k_{x_0}g^{-1},e] = [j^k_{x_0}g\cdot(j^k_{x_0}g^{-1},e)] = [j^k_{x_0}\mathrm{Id}, j^k_{x_0}g\cdot e]\ .$$

Conversely, if $E\to X$ is a natural bundle of order k, then we define a bundle isomorphism between E and the associated bundle to E_{x_0}:

$$\begin{array}{ccc} (\mathscr{U}^k\times E_{x_0})/G^k_{x_0} & = & E \\ [j^k_x\sigma, e_{x_0}] & \mapsto & \sigma_*^{-1}e_{x_0}\ . \end{array}$$

This is well defined: if we change the pair $(j^k_x\sigma, e_{x_0})$ by the equivalent element $j^k_{x_0}g\cdot(j^k_x\sigma,e_{x_0}) = (j^k_x(g\sigma), j^k_{x_0}g\cdot e_{x_0}) = (j^k_x(g\sigma), g_*e_{x_0})$, then:

$$[j^k_x(g\sigma), g_*e_{x_0}]\mapsto (g\sigma)_*^{-1}g_*e_{x_0} = \sigma_*^{-1}e_{x_0}\ .$$

□

Remark. The universal bundle can be simultaneously defined for all n-manifolds: given an n-manifold Z, let $\mathscr{U}^k_Z$ be the manifold formed by jets $j^k_z\sigma$ of diffeomorphisms $\sigma\colon Z \dashrightarrow X$ satisfying $\sigma(z)=x_0$. The universal bundle for Z is the projection $\mathscr{U}^k_Z\to Z$, $j^k_z\sigma\mapsto z$.

As a consequence, the functor "associated bundle" is defined for any n-manifold. This allows to extend the previous argument to obtain an equivalence of categories:

$$\begin{array}{ccc} G^k_{x_0}\text{-}\mathbf{Man} & = & \mathbf{Nat}^k_n \\ F_0 & \longmapsto & \mathfrak{F} \end{array}\quad ,\qquad \mathfrak{F}(Z) := (\mathscr{U}^k_Z\times F_0)/G^k_{x_0}\ .$$

As a consequence, the equivalence $\mathbf{Nat}^k_n = \mathbf{Nat}^k_X$ of Proposicion 2.1.1 follows.

Examples. Consider the standard linear representation of the linear group $G^1_{x_0} = Gl(T_{x_0}X)$ on $T_{x_0}X$. The corresponding natural bundle is the tangent bundle $TX\to X$.

More generally, the natural bundle corresponding to the linear representation of G^1_n in $\otimes^p T^*_{x_0}X\otimes^q T_{x_0}X$ is the vector bundle of (p,q)-tensors.

Consider the action of $G^1_{x_0} = Gl(T_{x_0}X)$ by translations on the quotient group $\mathbb{Z}/2\mathbb{Z} = Gl/Gl^o$. The corresponding natural bundle is the orientation covering $\overline{X}\to X$.

The group $G^k_{x_0}$ acts on $P^k_{x_0} := \{j^k_{x_0}f : f\in\mathscr{C}^\infty(X)\}$ as follows: $(j^k_{x_0}g)\cdot(j^k_{x_0}f) = j^k_{x_0}(f\circ g^{-1})$. The corresponding natural bundle is the bundle $J^k(X,\mathbb{R})\to X$ of k-jets of smooth functions on X.

Chapter 3

Homogeneous tensors

Let us fix a pseudo-Riemannian metric g and another tensor field ω; let R be the curvature tensor of g. A classical observation, probably due to Schouten ([42]), affirms that any tensor naturally constructed out of g and ω with differential operations can also be obtained using multilinear operations (tensor products, contractions) out of the tensors g, R, ω and the iterated covariant derivatives ∇R, $\nabla^2 R$,... $\nabla\omega$, $\nabla^2\omega$,...

Many authors have given more precise formulations or variations of this observation: Atiyah-Bott-Patodi ([6]), Epstein ([13]), Stredder ([47]), Krupka-Janyška ([26]), Slovák ([45]) or Kolář-Michor-Slovák ([24]), among many others.

Within this trend, in this Chapter we prove a formula that allows to determine the tensors T associated to g and ω, and satisfying an homogeneity condition of the form

$$T(\lambda^2 g, \lambda^a \omega) = \lambda^w T(g, \omega)\ , \quad \text{for all } \lambda > 0\ .$$

We make repeated use of particular cases of this formula in the rest of the thesis.

In the literature, the closest result to our statement is Stredder's ([47]). Nevertheless, as this author uses a different notion of naturalness, both results are not directly related to each other (see our comment in page 40).

3.1 Definition of natural tensors

Let $M \to X$ be the bundle of pseudo-Riemannian metrics on X with a fixed signature. As the rising and lowering of indices is an intrinsic operation, we can restrict our attention to the bundle $\otimes^p T^*X \to X$ of p-covariant tensors on X, with no loss of generality.

The sheaves of smooth sections of these bundles will be written, respectively,

$$Metrics \quad , \quad p\text{-}Tensors\ .$$

Definition. A natural p-tensor associated to a metric is a natural operator:

$$T\colon Metrics \longrightarrow p\text{-}Tensors\ ,$$

that is, a regular morphism of sheaves $T\colon Metrics \longrightarrow p\text{-}Tensors$ such that, for any diffeomorphism $\tau\colon U \to V$ between open sets of X, satisfies $T(\tau_* g) = \tau_* T(g)$.

The Einstein and the Lovelock tensors introduced in Chapters 4 and 6 are examples of natural 2-tensors.

The energy tensor of a k-form, introduced in page 69, is a natural tensor in a wider sense:

Definition. A natural p-tensor associated to a metric and a k-tensor is a natural operator:

$$T\colon Metrics \times k\text{-}Tensors \longrightarrow p\text{-}Tensors\ .$$

By abuse of language, tensors of the form $T(g,\omega)$, where g is a metric, ω is a k-tensor and $T\colon Metrics \times k\text{-}Tensors \longrightarrow p\text{-}Tensors$ is a natural operator, will also be called natural tensors.

3.2 Normal tensors

Let g be a germ of pseudo-Riemannian metric at a point $x \in X$, and let ∇ be its Levi-Civita connection.

The exponential map of ∇, $\exp_x\colon T_xX \to X$, is a diffeomorphism on a neighbourhood of the origin; hence, the corresponding neighbourhood of x inherits an affine structure:

Definition. The flat connection associated to ∇ around x is the linear connection $\overline{\nabla}$ corresponding, via the exponential map, to the canonical flat connection on T_xX (this construction depends on the point x).

A chart $(z_1,\dots,z_n)$ centred at x is a normal system for ∇ at the point x if, via the exponential map, it corresponds to a linear system of coordinates in T_xX.

Definition. Let A be a germ of p-covariant tensor at x. The r^{th}-normal tensor of A at the point x is defined as:

$$A_x^r := (\overline{\nabla}^r A)_x\ .$$

To our knowledge, these tensors were first introduced by Thomas ([49]).

If $(z_1,\dots,z_n)$ is a normal system centred at x, then the coefficients of A^r_x are:

$$A_{i_1\dots i_p,a_1\dots a_r} := \frac{\partial^r A_{i_1\dots i_p}}{\partial z_{a_1}\dots\partial z_{a_r}}(x) \tag{3.2.0.1}$$

As partial derivatives commute, the r^{th}-normal tensor of A at x lies in:

$$V_r := \otimes^p T^*_x X \otimes S^r T^*_x X$$

which is called the space of p-normal tensors of order r at x.

Let us write $g^r_x \in S^2 T^*_x X \otimes S^r T^*_x X$ for the r^{th}-normal tensor of the metric g at x. These tensors have additional symmetries that can be characterized using the following lemma (v.gr., [13]):

Gauss Lemma. *Let $(z_1,\dots,z_n)$ be a chart centred at $x \in X$. This chart is a normal system for ∇ if and only if the coefficients g_{ij} satisfy the equations:*

$$\sum_{j=0}^{n}(g_{ij} - g_{ij}(x))z_j = 0 \ .$$

Corollary 3.2.1. *The cyclic sum over the last $r+1$ indices of the normal tensor g^r_x is zero.*

Proof: Differentiating $r+1$ times the identity of the Gauss Lemma and evaluating at x, it follows:

$$g_{ij,k_1\dots k_r} + g_{ik_1,k_2\dots k_r j} + \dots + g_{ik_r,j\dots k_{r-1}} = 0 \, .$$

□

Definition. The vector space of metric normal tensors or order r at x is the vector subspace

$$N_r \subset S^2 T^*_x X \otimes S^r T^*_x X$$

of tensors such that the cyclic sum over their last $r+1$ indices is zero.

For $r=0$, $N_0 := M_x$ is defined to be the space of pseudo-Riemannian metrics at x of the fixed signature.

As an example, observe N_1 is always zero, for if $g^1_x \in N_1$, then:

$$g_{ij,k} = -g_{ik,j} = -g_{ki,j} = g_{kj,i} = g_{jk,i} = -g_{ji,k} = -g_{ij,k} \ .$$

Later on, we prove that any normal tensor $g_x^r \in N_r$ is the normal tensor associated to a metric (Lemma 3.3.4). Let us also remark that these vector spaces N_r are irreducible representations of the general linear group ([13]).

Lemma 3.2.2. *Metric normal tensors of order two have the following symmetries:*

1. *They are symmetric under the interchange of the first pair with the second pair of indices:*

$$g_{ij,kl} = g_{kl,ij} .$$

2. *The symmetrization of their last three indices is zero.*

Proof: Let us only check the first one, for the other is trivial. If $g_x^2 \in N_2$, then:

$$\begin{aligned} g_{ij,kl} &= -g_{il,jk} - g_{ik,lj} = -g_{li,jk} - g_{ki,jl} \\ &= g_{lk,ij} + g_{lj,ki} + g_{kl,ij} + g_{kj,li} \\ &= 2g_{kl,ij} + g_{jl,ki} + g_{kj,li} \\ &= 2g_{kl,ij} - g_{ji,lk} - g_{jk,il} + g_{kj,li} = 2g_{kl,ij} - g_{ij,kl} . \end{aligned}$$

□

Relation with the curvature and its covariant derivatives

The sequence $\{g_x^2, g_x^3, \dots, g_x^k\}$ of metric normal tensors at a point x, and the sequence of tensors $\{R_x, \nabla_x R, \dots, \nabla_x^{k-2} R\}$ formed by the curvature and its covariant derivatives at x, mutually determine each other.

The simplest case of this assertion is easily checked by direct computation:

Proposition 3.2.3 (Thomas)**.** *The second normal tensor and the Riemann-Christoffel tensor of a metric mutually determine each other, via the formulae:*

$$R_{ij,kl} = g_{il,jk} - g_{ik,jl} \quad , \quad g_{ij,kl} = \frac{-1}{3}\left(R_{iljk} + R_{ikjl}\right) .$$

Proof: In a chart where $g_{ij} = \mu_{ij}$ (the diagonal matrix with as many ± 1 as the fixed signature) and $g_{ij,k} = 0$, a direct computation shows:

$$R_{ij,kl} = \frac{-1}{2}\left(g_{ik,jl} - g_{il,jk} + g_{jl,ik} - g_{jk,il}\right) .$$

The first equality then follows because, in normal coordinates, $g_{ij,kl} = g_{kl,ij}$. The second equality follows from the first one, using the cyclic identity of g_x^2. □

In general, it is clear that $R_x, \dots, \nabla_x^{k-2} R$ can be written in terms of the derivatives of g at x in normal coordinates. To prove the converse, let us briefly recall an elegant computation, due to Atiyah-Bott-Patodi ([6]).

Let $x_1, \dots, x_n$ be a normal chart around $x \in X$, and let $H = x_i \partial_{x_i}$ be the field of homotheties in those coordinates (we use summation over repeated indices).

Let $D_1, \dots, D_n$ be the orthonormal basis of vector fields obtained by parallel transport of the orthonormal basis $(\partial_{x_1})_x, \dots, (\partial_{x_n})_x$ along the geodesics passing through x. Let $\theta^1, \dots, \theta^n$ be the dual basis of 1-forms and ω^i_j the connection 1-forms.

The geometric assumptions correspond to the following formulae:

$$i_H \theta^i = x^i \quad , \quad i_H \omega^i_j = 0 \quad , \quad g = \theta^i \otimes \theta^i \ .$$

Let a^i_j, b^i_j be the smooth functions relating the basis $\{\theta_i\}$ and $\{\mathrm{d}x_i\}$:

$$\theta^i = a^i_j \mathrm{d}x^j \quad , \quad \mathrm{d}x^j = b^j_k \theta^k \ .$$

In this normal chart, $g_{ij} = a^k_i a^k_j$. Hence, it is enough to prove that the Taylor expansion of R at x determines the Taylor expansion of the functions a^k_i at x.

Lemma 3.2.4. *It holds:*

$$(H^2 + H) a^i_l \ = \ a^i_\alpha b^\beta_j R^\alpha_{\beta kl} x^j x^k \ ,$$

and, as a consequence,

$$(n^2 + n)\, \widehat{a}^i_l[n] \ = \ x^j x^k \{\widehat{a^i_\alpha b^\beta_j R^\alpha_{\beta kl}}\}[n-2] \ ,$$

where $\widehat{f}[n]$ denotes the homogeneous component of degree n in the Taylor expansion of the smooth function f at the point x.

Proof: First of all, as the Levi-Civita connection is symmetric, we have $\mathrm{d}\theta^i = \omega^i_j \wedge \theta^j$, so:

$$H^L \theta^i = i_H \mathrm{d}\theta^i + \mathrm{d}x^i = i_H(\omega^i_j \wedge \theta^j) + \mathrm{d}x^i = -\omega^i_j x^j + \mathrm{d}x^i \ ,$$
$$H^L \theta^i = \left(H a^i_j + a^i_j \right) \mathrm{d}x^j \ .$$

For the curvature to appear, we need to differentiate again. To this end, observe that the second Cartan structural equation, for the frames $\{D_i\}$ and $\{\theta^j\}$, reads:

$$K^i_{jkl}\mathrm{d}x^k \wedge \mathrm{d}x^l = \mathrm{d}\omega^i_j - \omega^i_k \wedge \omega^k_j \ ,$$

for some smooth functions K^i_{jkl} related to the coefficients R^i_{jkl} of the curvature tensor, which is defined in terms of the $\{\partial_{x^i}\}$ and $\{\mathrm{d}x^j\}$, as follows:

$$2K^i_{jkl} = -a^i_\alpha b^\beta_j R^\alpha_{\beta kl} \ .$$

On the other hand, it will be useful to consider the function $r^2 = x^i x^i$. As $Hr = r$ it also holds:

$$H\left(\frac{x^i}{r}\right) = 0 \quad , \quad H^L\left(\frac{1}{r}\,\mathrm{d}x^i\right) = 0 \ ,$$

because these expressions are homogeneous of degree zero on r.

Now we can adequately differentiate twice the frame θ^i; on the one hand:

$$\begin{aligned} r\cdot H^L\cdot\frac{1}{r}\cdot H^L\theta^i &= r\cdot H^L\left(\frac{-x^j}{r}\,\omega^i_j\right) + 0 = -\frac{rx^j}{r}H^L\omega^i_j = -x^j\, i_H\mathrm{d}\omega^i_j = -x^j\, i_H\left(\mathrm{d}\omega^i_j - \omega^i_k\wedge\omega^k_j\right) \\ &= -x^j\, i_H\left(K^i_{jkl}\mathrm{d}x^k\wedge\mathrm{d}x^l\right) = -2x^jx^kK^i_{jkl}\mathrm{d}x^l = a^i_\alpha b^\beta_j R^\alpha_{\beta kl}x^jx^k\mathrm{d}x^l \ , \end{aligned}$$

and, on the other, this expression can also be computed in terms of the a^i_j:

$$\begin{aligned} r\cdot H^L\cdot\frac{1}{r}\cdot H^L\theta^i &= r\cdot H^L\left(\frac{Ha^i_j}{r}+\frac{a^i_j}{r}\right)\mathrm{d}x^j \\ &= r\left(\left(\frac{H^2a^i_j}{r}+\frac{Ha^i_j}{r}\right)\mathrm{d}x^j + \left(\frac{-Ha^i_j}{r}+\frac{-a^i_j}{r}\right)\mathrm{d}x^j + \left(\frac{Ha^i_j}{r}+\frac{a^i_j}{r}\right)\mathrm{d}x^j\right) \\ &= \left(H^2a^i_j + Ha^i_j\right)\mathrm{d}x^j \ . \end{aligned}$$

$\square$

3.3 Replacement Theorem

Let $T\colon \mathit{Metrics} \longrightarrow p\text{-}\mathit{Tensors}$ be a natural tensor of finite order.

On any coordinate chart $(x_1,\dots,x_n)$, the coefficients of $T(g)$ are

$$T(g)_{i_1\dots i_p} = F_{i_1\dots i_p}\left(g_{ij}, \frac{\partial g_{ij}}{\partial x_r}, \frac{\partial^2 g_{ij}}{\partial x_r \partial x_s}, \dots\right)$$

for certain smooth functions $F_{i_1\dots i_p}$.

Choosing normal coordinates $(z_1,\dots,z_n)$ for g at a point x, this formula becomes

$$T(g)_{i_1\dots i_p}(x) = F_{i_1\dots i_p}\left(g_{ij}(x), 0, \frac{\partial^2 g_{ij}}{\partial z_r \partial z_s}(x), \dots\right)$$

where, in this case, the functions $\frac{\partial^2 g_{ij}}{\partial z_r \partial z_s}(x)$ are the coefficients of the second normal tensor g^2_x, the functions $\frac{\partial^3 g_{ij}}{\partial z_r \partial z_s \partial z_t}(x)$ are the coefficients of the third normal tensor g^3_x,...

In other words, the natural tensor $T(g)$ is determined by a smooth function of the metric g and its normal tensors $(g^2_x, g^3_x, \dots)$ at a point x. This fact was called *Replacement Theorem* by Thomas.

The purpose of this Section is to prove a more complete formulation of this Replacement Theorem (see Theorem 3.3.7).

Quotients by the actions of Lie groups

Let $(X,\mathcal{O}_X)$ be a ringed space and let G be a group acting on X; that is, a group G together with a morphism of groups $G \to \operatorname{Aut} X$.

Definition. The quotient space $(X/G, \mathcal{O}_{X/G})$ is the ringed space defined as:

- The set X/G is endowed with the final topology of the quotient map $\pi\colon X \to X/G$.
- On any open set U of X/G, the structural sheaf is defined as:

$$\mathcal{O}_{X/G}(U) := \{f \in \mathcal{C}(U,\mathbb{R})\colon f \circ \pi \in \mathcal{O}_X(\pi^{-1}U)\}\ .$$

This quotient space $(X/G, \mathcal{O}_{X/G})$ has the corresponding universal property: for any given morphism of ringed spaces $\varphi\colon X \to Y$ such that $\varphi(g(x)) = \varphi(x)$, for any $x \in X, g \in G$, there exists a unique morphism of ringed spaces $\overline{\varphi}\colon X/G \to Y$ such that $\varphi = \overline{\varphi} \circ \pi$.

For what follows, we will need the following well-known results[1] regarding quotients of smooth manifolds by the action of Lie groups:

Proposition 3.3.1. *Let H be a closed, Lie subgroup of a Lie group G. Then, the quotient space G/H is a smooth manifold and the quotient map $\pi\colon G \to G/H$ is a submersion.*

Proposition 3.3.2. *Let G be a Lie group and let $G \times Z \to Z$ be a smooth action of G on a smooth manifold Z.*

If the action is transitive, then at any point $z \in Z$ the following diffeomorphism of smooth manifolds holds:

$$G/I_z = Z \quad , \quad [g] \mapsto g \cdot z$$

where I_z is the isotropy subgroup of z.

Let M_x be the smooth manifold of pseudo-Riemannian metrics with the fixed signature at a point $x \in X$.

The linear group $Gl_x := Gl(T_xX)$ acts on M_x and the isotropy subgroup of a metric g_x is the orthogonal group $O_{g_x} := O(T_xX, g_x)$.

As a particular case of the previous Proposition, it follows:

Corollary 3.3.3. *Fixing a metric $g_x \in M_x$, there exists a diffeomorphism:*

$$Gl_x/O_{g_x} \; = \; M_x \quad , \quad [\tau] \mapsto \tau(g_x)\,.$$

Reduction to normal form

Recall G_x^m denotes the Lie group of m-jets $j_x^m\tau$ of germs of diffeomorphisms of X leaving the point x fixed, and let $H_x^m := \{\tau \in G_x^m \, : \, j_x^1\tau \, = \, j_x^1(\mathrm{Id})\}$ be the subgroup of jets whose tangent linear map is the identity.

Let us recall that $M \to X$ denotes the bundle of pseudo-Riemannian metrics on X with a fixed signature. For any such a metric g, let us denote

$$(g_x^0 = g_x, g_x^1 = 0, g_x^2, g_x^3, \dots)$$

the sequence of its normal tensors at x.

[1] See, v. gr., N. Bourbaki, *Groupes et algèbres de Lie*, Masson, Paris (1982).

Lemma 3.3.4. *For any $m \geq 0$, the canonical map:*

$$J^m_x M \xrightarrow{\pi_m} N_0 \times \ldots \times N_m \quad , \quad j^m_x g \to (g^0_x, \ldots, g^m_x)$$

is a submersion, whose fibres are the orbits of H^m_x.

Therefore, it induces a diffeomorphism of smooth manifolds:

$$(J^m_x M)/H^m_x \; = \; N_0 \times \ldots \times N_m \; .$$

Proof: The maps π_m are G^m_x-equivariant, for they are canonically defined. Hence, the orbits of H^m_x are inside its fibres, because H^m_x acts by the identity on the spaces of normal tensors.

Conversely, let $j^m_x g, j^m_x \overline{g}$ be metric jets with the same normal tensors at x. Fix a basis on $T^*_x X$, and let $z_i, \overline{z}_i$ be the corresponding normal charts produced by g and $\overline{g}$; i.e., $\mathrm{d}_x z_i = \mathrm{d}_x \overline{z}_i$ is the chosen basis.

As g and $\overline{g}$ have the same normal tensors at x, it follows:

$$\frac{\partial g_{ij}}{\partial z^{i_1} \ldots \partial z^{i_l}}(x) = \frac{\partial \overline{g}_{ij}}{\partial \overline{z}^{i_1} \ldots \partial \overline{z}^{i_l}}(x)$$

for any $i_1 \ldots i_l$, with $l \leq m$.

The diffeomorphism $\tau(z_i) := \overline{z}_i$ belongs to H, because $\mathrm{d}_x z^i = \mathrm{d}_x \overline{z}^i$, and it satisfies:

$$\begin{aligned} \tau \cdot (j^m_x g) &= \tau \cdot \left(\sum_{i,j=1}^{n} \sum_{l=0}^{m} \sum_{i_1,\ldots,i_l=1}^{n} \frac{\partial g_{ij}}{\partial z^{i_1} \ldots \partial z^{i_l}}(x)\, z^{i_1} \ldots z^{i_l}\, \mathrm{d}_x z^i \otimes \mathrm{d}_x z^j \right) \\ &= \sum_{i,j=1}^{n} \sum_{l=0}^{m} \sum_{i_1,\ldots,i_l=1}^{n} \frac{\partial g_{ij}}{\partial z^{i_1} \ldots \partial z^{i_l}}(x)\, \overline{z}^{i_1} \ldots \overline{z}^{i_l}\, \mathrm{d}_x \overline{z}^i \otimes \mathrm{d}_x \overline{z}^j \; = \; j^m_x \overline{g} \; . \end{aligned}$$

To check π_m is a submersion, let us construct sections of π_m passing through any point of $J^m_x M$. To this end, fix a chart $(z_1, \ldots, z_n)$ centred at x.

Let $(T^0, \ldots, T^m) \in N_0 \times \ldots \times N_m$ be a sequence of normal tensors and let $T^s_{ij,a_1 \ldots a_s}$ be its components.

Consider the germ defined in these coordinates as follows:

$$g_m := \sum_{s=0}^{m} \left(\sum_{i,j=1}^{n} \sum_{a_1 \ldots a_s = 1}^{n} T^s_{ij,a_1 \ldots a_s} z^{a_1} \ldots z^{a_s} \right) \mathrm{d}z^i \otimes \mathrm{d}z^j$$

The (global) section s_m, that depends on the chosen coordinates, is defined as:

$$s_m(T^0,\dots,T^m) := j^m_x g_m \ .$$

The symmetries of the tensors $T^0,\dots,T^r$ guarantee that $(z_1,\dots,z_n)$ is a normal system for g_m at x, and therefore, the normal tensors of $j^m_x g_m$ at x are $T^0,\dots,T^m$.

The section s_m can pass through any prefixed point $j^m_x g$, by choosing $(z_1,\dots,z_n)$ to be a normal system for g. □

An extension: pair of metric and k-tensor

Later on, we will need a more general version of this Lemma.

Let $T_k := \otimes^k T^*X$ denote the bundle of k-covariant tensors on X, and let:

$$J^m_x M \times J^m_x T_k \xrightarrow{\pi_m} \prod_{r=0}^m N_r \times \prod_{s=0}^m V_s \quad , \quad (j^m_x g, j^m_x \omega) \to (g^0_x,\dots,\omega^m_x)$$

be the map sending a pair of m-jets to their normal tensors.

An analogous reasoning to that of Lemma 3.3.4 proves:

Lemma 3.3.5. *For any $m \geq 0$, the canonical map:*

$$J^m_x M \times J^m_x T_k \xrightarrow{\pi_m} \prod_{r=0}^m N_r \times \prod_{s=0}^m V_s \quad , \quad (j^m_x g, j^m_x \omega) \to (g^0_x,\dots,\omega^m_x)$$

is a submersion, whose fibres are the orbits of H^m_x.

Therefore, it induces a diffeomorphism of smooth manifolds:

$$(J^m_x M \times J^m_x T_k)/H^m_x \;=\; N_0 \times \ldots \times V_m \ .$$

Let G^∞_x be the inverse limit of the G^m_x and let $H^\infty_x \subset G^\infty_x$ be the group of jets of diffeomorphisms whose tangent linear map at x is the identity.

The smooth sections s_m constructed as in the proof of Theorem 3.3.4 make the following squares commutative:

$$\begin{array}{ccc} \prod\limits_{r=0}^{m} N_r \times \prod\limits_{s=0}^{m} V_s & \xrightarrow{s_m} & J_x^m M \times J_x^m T_k. \\ \downarrow & & \downarrow \\ \prod\limits_{r=0}^{m-1} N_r \times \prod\limits_{s=0}^{m-1} V_s & \xrightarrow{s_{m-1}} & J_x^{m-1} M \times J_x^{m-1} T_k \end{array} \quad .$$

Taking inverse limits, it follows:

Lemma 3.3.6. *The fibres of the canonical map:*

$$J_x^\infty M \times J_x^\infty T_k \xrightarrow{\pi_\infty} \prod_{r=0}^{\infty} N_r \times \prod_{s=0}^{\infty} V_s \quad , \quad (j_x^\infty g, j_x^\infty \omega) \to (g_x^0, \ldots, \omega_x^0, \ldots)$$

are the orbits of H_x^∞.

Moreover, there exist global sections of π_∞ *passing through any point of* $J_x^\infty M \times J_x^\infty T_k$. *Therefore, this map* π_∞ *induces an isomorphism of ringed spaces:*

$$(J_x^\infty M \times J_x^\infty T_k)/H_x^\infty = \prod_{r=0}^{\infty} N_r \times \prod_{s=0}^{\infty} V_s \ .$$

Remark. As a consequence of this isomorphism and the exact sequence

$$0 \longrightarrow H_x^\infty \longrightarrow G_x^\infty \longrightarrow Gl_x \longrightarrow 0 \ ,$$

the map π_∞ also induces a bijection:

$$\begin{array}{c} \{\, G_x^\infty\text{-equivariant smooth maps } T_x \colon J_x^\infty M \times J_x^\infty T_k \longrightarrow \otimes^p T_x^* X \,\} \\ \Big\| \\ \{\, Gl_x\text{-equivariant smooth maps } \bar{\mathrm{t}} \colon \prod\limits_{r=0}^{\infty} N_r \times \prod\limits_{s=0}^{\infty} V_s \longrightarrow \otimes^p T_x^* X \,\} \end{array} \quad .$$

Theorem 3.3.7. *Let* $x \in X$ *be any point. There exists an isomorphism of* $\mathbb{R}$*-vector spaces:*

$$\begin{array}{c} \{\, \textit{Natural tensors } T \colon Metrics \times k\text{-}Tensors \longrightarrow p\text{-}Tensors \,\} \\ \Big\| \\ \{\, Gl_x\textit{-equivariant smooth maps } \bar{\mathrm{t}} \colon \prod\limits_{r=0}^{\infty} N_r \times \prod\limits_{s=0}^{\infty} V_s \longrightarrow \otimes^p T_x^* X \,\} \end{array} \quad .$$

The relation between a natural tensor T and the corresponding smooth map $\bar{\mathfrak{t}}$ is given by the formula:

$$T(g,\omega)_x = \bar{\mathfrak{t}}(g^0_x, g^2_x, \ldots; \omega^0_x, \omega^1_x, \ldots)$$

where $(g^0_x, g^2_x, \ldots; \omega^0_x, \omega^1_x, \ldots)$ is the sequence of normal tensors of g and ω at x.

Proof: Due to Proposition 2.3.3 and the remark above, the map of the statement establishes bijections:

$$\begin{array}{c} \{\text{ Natural tensors } T\colon Metrics \times k\text{-}Tensors \longrightarrow p\text{-}Tensors \ \} \\ \Big\| \\ \{\ G^\infty_x\text{-equivariant smooth maps } T_x : J^\infty_x M \times J^\infty_x T_k \longrightarrow \otimes^p T^*_x X \ \} \\ \Big\| \\ \{\ Gl_x\text{-equivariant smooth maps } \bar{\mathfrak{t}}\colon \prod\limits_{r=0}^{\infty} N_r \times \prod\limits_{s=0}^{\infty} V_s \longrightarrow \otimes^p T^*_x X \ \} \end{array}.$$

As these spaces consist on maps taking values on vector spaces, they inherit a linear structure, and these bijections are compatible with this linear structure. □

Once a metric g_x is fixed at the point x, the diffeomorphism $(Gl_x/O_{g_x}) = M_x$ of Corollary 3.3.3 makes it possible to substitute the linear group by the orthogonal group.

Theorem 3.3.8. *Let $x \in X$ be a point and let $g_x \in M_x$ be a pseudo-Riemannian metric at x.*

There exists an $\mathbb{R}$-linear isomorphism:

$$\begin{array}{c} \{\ Natural\ tensors\ T\colon Metrics \times k\text{-}Tensors \longrightarrow p\text{-}Tensors \ \} \\ \Big\| \\ \{\ O_{g_x}\text{-}equivariant\ smooth\ maps\ \mathfrak{t}\colon \prod\limits_{r=2}^{\infty} N_r \times \prod\limits_{s=0}^{\infty} V_s \longrightarrow \otimes^p T^*_x X \ \} \end{array}.$$

On any metric g having the value g_x at x, the relation between a natural tensor T and the corresponding smooth map $\mathfrak{t}$ is given by the formula:

$$T(g,\omega)_x = \mathfrak{t}(g^2_x, g^3_x, \ldots; \omega^0_x, \omega^1_x, \ldots) .$$

Proof: Once a metric g_x is fixed, we only have to prove the isomorphism between the vector space of Gl_x-equivariant smooth maps

$$N_0 \times \prod_{r=2}^{\infty} N_r \times \prod_{s=0}^{\infty} V_s \xrightarrow{\ \bar{\mathfrak{t}}\ } \otimes^p T^*_x X$$

and that of O_{g_x}-equivariant smooth maps:

$$\prod_{r=2}^{\infty} N_r \times \prod_{s=0}^{\infty} V_s \xrightarrow{\ \mathfrak{t}\ } \otimes^p T^*_x X \ .$$

Given such an $\bar{\mathfrak{t}}$, the corresponding O_{g_x}-equivariant map is obtained by restriction:

$$\begin{tikzcd}
\prod_{r=0}^{\infty} N_r \times \prod_{s=0}^{\infty} V_s \arrow[r, "\bar{\mathfrak{t}}"] & \otimes^p T^*_x X. \\
\{g_x\} \times \prod_{r=2}^{\infty} N_r \times \prod_{s=0}^{\infty} V_s \arrow[u] \arrow[ur, "\mathfrak{t}"'] &
\end{tikzcd}$$

Conversely, if $\mathfrak{t}$ is a O_{g_x}-equivariant map, we first define:

$$\widetilde{\mathfrak{t}} \colon Gl_x \times \prod_{r=2}^{\infty} N_r \times \prod_{s=0}^{\infty} V_s \longrightarrow \otimes^p T^*_x X \qquad , \qquad \widetilde{\mathfrak{t}}(\tau, \_, \_) := \tau \left[f\left(\tau^{-1}\_, \tau^{-1}\_ \right) \right] \ ,$$

that satisfies $\widetilde{\mathfrak{t}}(\tau \cdot \sigma^{-1}, \sigma\_, \sigma\_) = \widetilde{\mathfrak{t}}(\tau, \_, \_)$ for any $\sigma \in O_{g_x}$.

Hence, it factors through the quotient by the action of O_{g_x} (recall that $Gl_x / O_{g_x} = N_0$, by Corollary 3.3.3):

$$N_0 \times \prod_{r=2}^{\infty} N_r \times \prod_{s=0}^{\infty} V_s \xrightarrow{\ \mathfrak{t}\ } \otimes^p T^*_x X \ .$$

□

Corollary 3.3.9. *Let $x \in X$ be a point and let $g_x \in M_x$ be a pseudo-Riemannian metric at x.*

There exists an $\mathbb{R}$-linear isomorphism:

$$\begin{array}{c}
\{ \textit{Natural tensors } T \colon \textit{Metrics} \longrightarrow p\textit{-Tensors of order } \leq k \ \} \\
\Big\| \\
\{ \ O_{g_x}\textit{-equivariant smooth maps } \mathfrak{t} \colon N_2 \times \ldots \times N_k \longrightarrow \otimes^p T^*_x X \ \}
\end{array} .$$

Remark. A natural tensor $T \colon \textit{Metrics} \longrightarrow p\textit{-Tensors}$ of finite order is *algebraic* (or polynomial) if, in any coordinate chart $(x_1, \ldots, x_n)$, the coefficients of $T(g)_{i_1 \ldots i_p}$ are universal polynomials in the variables

$$\left(g_{ij}, \frac{\partial g_{ij}}{\partial x_r}, \frac{\partial^2 g_{ij}}{\partial x_r \partial x_s}, \ldots ; \left(det\, g_{ij}\right)^{-1} \right) .$$

In this context and for Riemannian metrics, Stredder ([47]) proved a similar result to Corollary 3.3.9 that can be reformulated as follows:

Theorem 3.3.10. *Let $x \in X$ be a point and let $g_x \in M_x$ be a pseudo-Riemannian metric at x. There exists an $\mathbb{R}$-linear isomorphism:*

$$\begin{array}{c} \{\, Algebraic\ natural\ tensors\ \ T\colon Metrics \longrightarrow p\text{-}Tensors\ \ of\ order\ \leq k \,\} \\ \Big\| \\ \{\, O_{g_x}\text{-}equivariant\ polynomial\ maps\ \ \mathfrak{t}\colon N_2 \times \ldots \times N_k \longrightarrow \otimes^p T_x^* X \,\} \end{array} .$$

This statement can be proved in a similar manner to Theorem 3.3.8, but considering algebraic maps instead of smooth maps at each step. In particular, it requires to study the relation between the algebraic structures on Gl_x/O_{g_x} and M_x (see Appendix I of [6]).

3.4 Homogeneous natural tensors

Definition. A natural tensor $T\colon Metrics \longrightarrow p\text{-}Tensors$ is homogeneous of weight $w \in \mathbb{R}$ if for any metric g and any real number $\lambda > 0$, it holds:

$$T(\lambda^2 g) = \lambda^w \, T(g) .$$

If $w = 0$, we say T is independent of the unit of scale.

The Ricci tensor or the Einstein tensor are examples of natural 2-tensor independent of the unit of scale.

The energy tensor $T(g,\omega)$ associated to a metric g and a k-form ω, (see page 69), is a natural tensor that, for any real number $\lambda > 0$, satisfies the homogeneity condition:

$$T(\lambda^2 g, \lambda^{k-1}\omega) = T(g,\omega) .$$

Definition. A natural tensor $T\colon Metrics \times k\text{-}Tensors \longrightarrow p\text{-}Tensors$ is homogeneous of relative weight $(a;w)$, with $a \in \mathbb{Z}$ and $w \in \mathbb{R}$, if, for any metric g, any $(k,0)$-tensor ω and any real number $\lambda > 0$, it holds:

$$T(\lambda^2 g, \lambda^a \omega) = \lambda^w \, T(g,\omega) \, .$$

Let $T(g,\omega)$ be a natural p-tensor and let:

$$\prod_{r=2}^{\infty} N_r \times \prod_{s=0}^{\infty} V_s \xrightarrow{\ \mathfrak{t}\ } \otimes^p T_x X$$

be the associated O_{g_x}-equivariant map.

Lemma 3.4.1. *The condition of $T(g,\omega)$ being homogeneous of relative weight $(a;w)$ is equivalent to the following homogeneity condition on $\mathfrak{t}$, for all $\lambda > 0$:*

$$\mathfrak{t}(\lambda^2 g_x^2, \ldots, \lambda^r g_x^r, \ldots, \lambda^{k-a} \omega_x^0, \ldots, \lambda^{k+s-a} \omega_x^s, \ldots) = \lambda^{p-w} \, \mathfrak{t}(g_x^2, \ldots, \omega_x^0, \ldots) \, .$$

Proof: Let τ denote the local homothety of ratio λ on some coordinates centred at x, so that τ_* is the homothety of ratio λ^p when acting on p-tensors.

First of all, let us check

$$T(\tau_*(\lambda^{-2} g), \tau_*(\lambda^{-a}\omega))_x = \mathfrak{t}(\lambda^2 g_x^2, \ldots, \lambda^r g_x^r, \ldots, \lambda^{k-a} \omega_x^0, \ldots, \lambda^{k+s-a} \omega_x^s, \ldots) \, .$$

To see this, we have to prove that the normal tensors associated to $\tau_*(\lambda^{-2} g)$ and $\tau_*(\lambda^{-a}\omega)$ at x are

$$g_x^0, \lambda^2 g_x^2, \ldots, \lambda^r g_x^r, \ldots \qquad , \qquad \lambda^{k-a}\omega_x^0, \ldots, \lambda^{k+s-a}\omega_x^s, \ldots$$

that is due to the fact that normal tensors are natural:

$$\left[\tau_*(\lambda^{-2} g)\right]_x^r = \tau_*\left(\left[\lambda^{-2} g\right]_x^r\right) = \tau_*\left(\lambda^{-2} g_x^r\right) = \lambda^{r+2}\lambda^{-2} g_x^r = \lambda^r g_x^r ,$$

and analogously for $\tau_*(\lambda^{-a}\omega)$.

On the other hand,

$$T(\tau_*(\lambda^{-2} g), \tau_*(\lambda^{-a}\omega))_x = \tau_*\left[T(\lambda^{-2} g, \lambda^{-a}\omega)\right]_x = \lambda^p \, T(\lambda^{-2} g, \lambda^{-a}\omega)_x$$

that amounts to

$$\lambda^{p-w} \, T(g,\omega)_x = \lambda^{p-w} \, \mathfrak{t}(g_x^2, \ldots, \omega_x^0, \ldots)$$

if and only if T is homogeneous of relative weight $(a;w)$. □

The following theorem is borrowed from the book by Kolář-Michor-Slovák ([24], 24.1), where the statement is presented for a finite sequence of vector spaces. Our generalization to a countable family is straightforward.

Homogeneous Function Theorem: *Let $\{E_i\}_{i\in\mathbb{N}}$ be finite dimensional vector spaces.*

Let $f : \prod_{i=1}^{\infty} E_i \to \mathbb{R}$ be a smooth function such that there exist positive real numbers $a_i > 0$, and $w \in \mathbb{R}$ satisfying:

$$f(\lambda^{a_1} e_1, \ldots, \lambda^{a_i} e_i, \ldots) = \lambda^w f(e_1, \ldots, e_i, \ldots) \tag{3.4.0.2}$$

for any positive real number $\lambda > 0$ and any $(e_1, \ldots, e_i, \ldots) \in \prod_{i=1}^{\infty} E_i$.

Then, f depends on a finite number of variables $e_1, \ldots, e_k$ and it is a sum of monomials of degree d_i in e_i satisfying the relation

$$a_1 d_1 + \cdots + a_k d_k = w\ . \tag{3.4.0.3}$$

If there are no natural numbers $d_1, \ldots, d_r \in \mathbb{N} \cup \{0\}$ satisfying this equation, then f is the zero map.

In other words, for any finite dimensional vector space W, there exists an $\mathbb{R}$-linear isomorphism:

$$\left[\text{Smooth maps } f : \prod_{i=1}^{\infty} E_i \to W \text{ satisfying (3.4.0.2)}\right]$$
$$\Big\|$$
$$\bigoplus_{d_1,\ldots,d_k} \mathrm{Hom}_{\mathbb{R}}(S^{d_1}E_1 \otimes \ldots \otimes S^{d_k}E_k, W)$$

where $d_1, \ldots, d_k$ run over the non-negative integers solutions of (3.4.0.3).

Proof: Firstly, observe $w \geq 0$ because, otherwise, (3.4.0.2) is contradictory when $\lambda \to 0$.

As f is smooth, there exists a neighbourhood $U = \{|e_1| < \epsilon, \ldots, |e_k| < \epsilon\} \subset \prod_{i=1}^{\infty} E_i$ of the origin and a smooth map $\overline{f} : \pi_k(U) \to \mathbb{R}$ such that $f_{|U} = (\overline{f} \circ \pi_k)_{|U}$.

As the $a_1, \ldots, a_k$ are positive, there exist a neighbourhood of zero, $\overline{V}_0 \subset \mathbb{R}$, and a neighbourhood of the origin $V \subset \pi_k(U)$, such that for any $(e_1, \ldots, e_k) \in V$, and any $\lambda \in \overline{V}_0$ positive, the vector $(\lambda^{a_1} e_1, \ldots, \lambda^{a_k} e_k)$ lies in V.

On that neighbourhood V, the function $\overline{f}$ satisfies the homogeneity condition:

$$\overline{f}(\lambda^{a_1} e_1, \ldots, \lambda^{a_k} e_k) = \lambda^w \overline{f}(e_1, \ldots, e_k) \tag{3.4.0.4}$$

for any positive real number $\lambda \in \overline{V}_0$.

Differentiating this equation, we obtain analogous conditions for the partial derivatives of $\overline{f}$; v.gr.:

$$\frac{\partial \overline{f}}{\partial x_1}(\lambda^{a_1}e_1,\dots,\lambda^{a_k}e_k) = \lambda^{w-a_1}\frac{\partial \overline{f}}{\partial x_1}(e_1,\dots,e_k)\ .$$

If the order of derivation is big enough, the corresponding partial derivative is homogeneous of negative weight, and hence zero. This implies that $\overline{f}$ is a polynomial; the homogeneity condition (3.4.0.4) is then satisfied for any positive $\lambda \in \overline{V}_0$ if and only if its monomials satisfy (3.4.0.3).

Finally, given any $e = (e_1,\dots,e_n,\dots) \in \prod_{i=1}^{\infty} E_i$, take $\lambda \in \mathbb{R}^+$ such that $(\lambda^{a_1}e_1,\dots,\lambda^{a_k}e_k,\dots)$ lies in U. Then:

$$f(e) = \lambda^{-w} f(\lambda^{a_1}e_1,\dots,\lambda^{a_n}e_n,\dots) = \lambda^{-w}\overline{f}(\lambda^{a_1}e_1,\dots,\lambda^{a_k}e_k) = \overline{f}(e_1,\dots,e_k)$$

and f only depends on the first k variables.

□

Theorem 3.4.2. *Let $x \in X$ be a point and let g_x be a pseudo-Riemannian metric at x.*

There exists an $\mathbb{R}$-linear isomorphism:

$$\left[\begin{array}{c} \textit{Natural tensors } T: \mathit{Metrics} \times k\text{-}\mathit{Tensors} \longrightarrow p\text{-}\mathit{Tensors} \\ \textit{homogeneous of relative weight } (a;w), \textit{ with } a<k \end{array}\right]$$
$$\Big\|$$
$$\bigoplus_{d_i,\overline{d}_j} \mathrm{Hom}_{O_{g_x}}\left(S^{d_2}N_2 \otimes \cdots \otimes S^{d_r}N_r \otimes S^{\overline{d}_0}V_0 \otimes \ldots \otimes S^{\overline{d}_s}V_s\ ,\ \otimes^p T_x^*X\right)$$

where the summation is over all sequences $\{d_2,\dots,d_r\}$ and $\{\overline{d}_0,\dots,\overline{d}_s\}$, of non-negative integers satisfying:

$$2d_2 + \ldots + r\,d_r + (k-a)\overline{d}_0 + \ldots + (k+s-a)\overline{d}_s = p - w\ . \qquad (3.4.0.5)$$

If this equation has no solutions, the above vector space reduces to zero.

Proof: It is an immediate consequence of Theorem 3.3.8, using Lemma 3.4.1 and the Homogeneous Function Theorem above.

□

Remark. The O_{g_x}-equivariant linear maps that appear in the theorem can be explicitly computed using the isomorphism:

$$\mathrm{Hom}_{O_{g_x}}(E,F) = \mathrm{Hom}_{O_{g_x}}\left(E \otimes F^*, \mathbb{R}\right)$$

and applying the Main Theorem for the orthogonal group ((Theorem A.0.5), which says that the vector space $\mathrm{Hom}_{O_{g_x}}\left(\otimes^r T_x^* X, \mathbb{R}\right)$ is spanned by iterated contraction of indices:

$$e_1 \otimes \ldots \otimes e_r \longmapsto g(e_{\sigma(1)}, e_{\sigma(2)}) \cdot \ldots \cdot g(e_{\sigma_{r-1}}, e_{\sigma(r)})$$

where σ is a permutation of $1, \ldots, r$.

This procedure will systematically applied in the following three Chapters.

Corollary 3.4.3. *Any natural tensor $T : Metrics \times k\text{-}Tensors \longrightarrow p\text{-}Tensors$ homogeneous of relative weight $(a;w)$, with $a < k$, has finite order.*

Moreover, w is an integer number and the vector space of such tensors is finite dimensional.

As a corollary, it also follows a remarkable characterization of the Levi-Civita connection, due to Epstein ([13]):

Corollary 3.4.4. *The only linear connection ∇_g, naturally associated to a pseudo-Riemannian metric g, that is independent of the unit of scale (i.e., $\nabla_{\lambda^2 g} = \nabla_g$) is the Levi-Civita connection.*

Proof: Any other such linear connection $\overline{\nabla}_g$ differs from the Levi-Civita connection ∇_g in a 3-tensor of weight 2: $T(D_1, D_2, D_3) := D_3 \cdot \left(\nabla_{D_1} D_2 - \overline{\nabla}_{D_1} D_2\right)$. As equation (3.4.0.5) has no solutions in this case, that tensor has to be zero.

□

Remark. In presence of an orientation, there exists a similar result to Theorem 3.4.2, but replacing the orthogonal group O_{g_x} by the special orthogonal group SO_{g_x}.

If $Orient$ denotes the sheaf of orientations on X, a natural operator:

$$T : Metrics \times Orient \longrightarrow p\text{-}Tensors$$

is said homogeneous of relative weight $(a;w)$ if it satisfies:

$$T(\lambda^2 g, \lambda^a \omega, or) = \lambda^w T(g, \omega, or) \,, \quad \forall\ \lambda \in \mathbb{R}^+ \ .$$

An analogous reasoning to that proving Theorem 3.4.2 shows that, if $x \in X$ is a point and

g_x is a pseudo-Riemannian metric at x, then there exists an $\mathbb{R}$-linear isomorphism:

$$\left[\begin{array}{c}\text{Natural tensors } T\colon \mathit{Metrics} \times k\text{-}\mathit{Tensors} \times \mathit{Orient} \longrightarrow p\text{-}\mathit{Tensors} \\ \text{homogeneous of relative weight } (a;w)\end{array}\right]$$
$$\|$$
$$\bigoplus_{d_i,\bar{d}_j} \operatorname{Hom}_{SO_{g_x}}\left(S^{d_2}N_2 \otimes \cdots \otimes S^{d_r}N_r \otimes S^{\bar{d}_0}V_0 \otimes \ldots \otimes S^{\bar{d}_s}V_s\ ,\ \otimes^p T_x^*X\right)$$

where the summation is over all sequences $\{d_i\}$ of non-negative integers fulfilling (3.4.0.5).

Thus, the use of Theorem A.0.5 also allows, in certain cases, an explicit computation of the vector spaces under consideration.

Chapter 4

Characterization of the Einstein tensor

In General Relativity, it is supposed a field equation of the following type:

$$G(g) = T$$

where T is the energy-momentum 2-covariant tensor of the matter distribution and $G(g)$ is a suitable natural tensor associated to the time metric g of space-time.

The infinitesimal conservation of mass-energy is encoded in the equation $\operatorname{div} T = 0$, so one is forced to choose for the left-hand side of this equation a natural tensor $G(g)$ that is divergence-free.

On the other hand, the dimensional analysis on Newtonian gravitation shows that the 2-covariant matter tensor T is independent of the unit of time. In General Relativity, the time metric g measures proper time, so that changing the unit of time amounts to replacing g by $\lambda^2 g$. Hence, a tensor $G(g)$ is said independent of the unit of scale if it satisfies:

$$G(\lambda^2 g) = G(g) \quad , \quad \forall\ \lambda \in \mathbb{R}^+ \ . \tag{4.0.0.1}$$

The main result of this Chapter (Corollary 4.2.3) characterizes the Einstein tensor of a pseudo-Riemannian metric as the only natural 2-tensor $G(g)$ which is divergence-free and independent of the unit of scale; i.e., satisfies condition (4.0.0.1). In contrast to other classical results, this characterization is valid in any dimension, there is no symmetry hypothesis and the dependence of $G(g)$ is not even assumed to be through derivatives of the metric g.

We also compute all natural 2-tensors $G(g)$ with zero divergence and that are homogeneous of weight greater than -2 (i.e., satisfying $G(\lambda^2 g) = \lambda^w G(g)$, for some $w > -2$). Once again, the Einstein tensor is essentially the only possibility (Theorem 4.2.2). In General Relativity, the meaning of this result is that is not possible to find a field equation, alternative to Einstein's equation, that give rise, in the newtonian approximation, to a law of universal gravitation different from the traditional inverse-square force law (see our comment in page 52).

In the last Section of the Chapter, we extend these ideas to a Weyl spacetime.

Part of the original results of this Chapter have been published in [36].

4.1 The matter energy-momentum tensor

To begin with, let us briefly comment on the definition of the energy-momentum tensor T of the matter distribution in a relativistic spacetime.

Throughout this Section, a space-time (X, g, ω_X) is an oriented Lorentz manifold of dimension 4; that is, X is a smooth manifold endowed with a non-singular metric g of signature $(+,-,-,-)$, called the time metric, and with a volume form ω_X. If necessary, we will also assume X is time-oriented; i.e., that there is a fixed equivalence class of timelike vector fields.

4.1.1 Representation of matter

Definition. A particle is an oriented smooth curve $S \subset X$, called trajectory, with a future pointing tangent vector field I, called its impulse, of constant modulus $m \geq 0$, called mass.

The representation of a fluid of particles is achieved describing the impulse on any infinitesimal region of spacetime.

Indeed, it is enough to specify the impulse crossing each 3-dimensional region, since trajectories are curves and any such a region will intersect the trajectory of some particles of the fluid at single points. The impulse crossing a small oriented 3-dimensional region is defined to be the sum of the impulses of the particles intersecting it, with sign +1 if the impulse points outside the oriented region, sign 0 if it is tangent and sign -1 if it points inside. Moreover, as these regions are infinitesimal, it is enough to consider the case of parallelograms.

Summing up, the representation of a continuum distribution of matter is accomplished by means of a a vector-valued 3-form, Π_3, with the following intuitive interpretation:

$$\Pi_3(D_1, D_2 D_3) = \left[\begin{array}{c} \text{Sum with sign of the impulses of the particles crossing} \\ \text{the infinitesimal oriented parallelogram } (D_1, D_2, D_3) \end{array} \right]$$

for any vectors $D_1, D_2, D_3 \in T_x X$, the sign of the impulse I of a particle being the sign of $(\omega_X)_x(I, D_1, D_2, D_3)$.

Definition. A continuum distribution of particles on spacetime is represented by a 3-form Π_3 on X with values on TX, called the impulse form of the fluid.

Example. In a *dust*, close particles are assumed to have the same velocity, so that the motion of near particles is defined by some future pointing, unitary vector field U.

The idea is that any integral curve of U represents the motion of a small spatial region, so that the mass density may change form one region to another.

To write down the impulse form, let (D_1, D_2, D_3) be an orthonormal basis of space-like vector fields, and let $(\theta_0, \theta_1, \theta_2, \theta_3)$ be the dual basis of (U, D_1, D_2, D_3).

Since the impulse of any particle is proportional to U, we have:

$$\Pi_3(D_1, D_2, D_3) = \rho U$$

where ρ is the mass contained in the unit cube (D_1, D_2, D_3); i.e., ρ is the mass density.

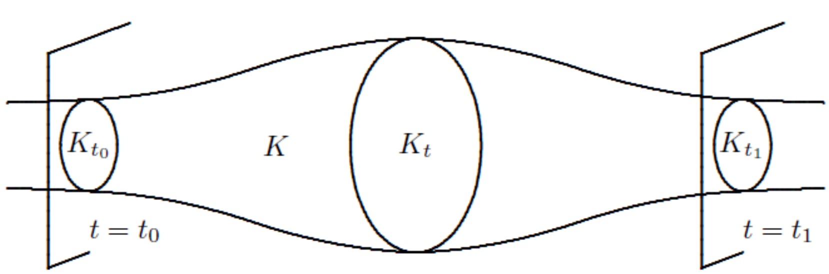
On the other hand, $\Pi_3(U, D_i, D_j) = 0$, because in this case the impulse is tangent to the considered parallelogram.

Since $\omega_X = \theta_0 \wedge \theta_1 \wedge \theta_2 \wedge \theta_3$, the impulse 3-form of a dust of mass density ρ and velocity U is defined to be:

$$\Pi_3 = (\theta_1 \wedge \theta_2 \wedge \theta_3) \otimes (\rho U) = \rho(i_U \omega_X) \otimes U \ .$$

Infinitesimal conservation of impulse

This sign convention in the definition of Π_3 is adopted so that the following interpretation holds: let $(X, g) = (\mathbb{R}^4, \mathrm{d}t^2 - \mathrm{d}x_1^2 - \mathrm{d}x_2^2 - \mathrm{d}x_3^2)$ be the Minkowski spacetime and consider a spatially compact tube K between two instants t_0, t_1.



If we consider on its boundary ∂K the orientation needed to apply Stoke's theorem, then:

$$\int_K d_\nabla \Pi_3 = \int_{\partial K} \Pi_3 = \begin{bmatrix} \text{Impulse in} \\ K_{t_1} \end{bmatrix} - \begin{bmatrix} \text{Impulse in} \\ K_{t_0} \end{bmatrix} + \int_{\text{lateral}} \Pi_3 \ ,$$

where the signs on the first two addends appear because in K_{t_1} (resp. K_{t_0}) the impulse of any particle points outside (inside) K.

In the lateral side of the tube, if the impulse points outside, then it is a particle going out of K whereas if it points inside, then it is a particle going into K. Hence,

$$\int_K d_\nabla \Pi_3 = \begin{bmatrix} \text{Impulse in} \\ K_{t_1} \end{bmatrix} - \begin{bmatrix} \text{Impulse in} \\ K_{t_0} \end{bmatrix} + \begin{bmatrix} \text{Impulse} \\ \text{going out} \end{bmatrix} - \begin{bmatrix} \text{Impulse} \\ \text{going in} \end{bmatrix} .$$

Therefore, the condition $d_\nabla \Pi_3 = 0$ amounts to the following *infinitesimal conservation of impulse*:

$$\begin{bmatrix} \text{Impulse in} \\ K_{t_1} \end{bmatrix} - \begin{bmatrix} \text{Impulse in} \\ K_{t_0} \end{bmatrix} = \begin{bmatrix} \text{Impulse} \\ \text{going in} \end{bmatrix} - \begin{bmatrix} \text{Impulse} \\ \text{going out} \end{bmatrix} .$$

Energy-momentum tensors

In the literature, the standard procedure to describe the impulse of a fluid is via its energy-momentum tensor. This tensor is related with the impulse form via the isomorphism of Corollary 4.1.2. But first, let us prove a general version of this statement, that will also be used in Chapter 5.

Let $E \to X$ be a vector bundle endowed with a linear connection ∇'.

The pair of connections ∇, ∇' on TX and E allow to define a linear connection $\widetilde{\nabla}$ on $TX \otimes E$, whose associated differential is:

$$d_{\widetilde{\nabla}} : TX \otimes E \rightsquigarrow T^*X \otimes TX \otimes E \quad , \quad D \otimes e \mapsto (d_\nabla D) \otimes e + D \otimes d_{\nabla'} e \ .$$

Definition. The divergence of a vector field J with values on E is the contraction of the first covariant and first contravariant indices of $d_{\widetilde{\nabla}} J$:

$$\mathrm{div}_{\widetilde{\nabla}} : \ TX \otimes E \ \rightsquigarrow \ E \quad , \quad J \mapsto c_1^1(d_{\widetilde{\nabla}} J) \ .$$

If E is a bundle of tensors, this operator $\mathrm{div}_{\widetilde{\nabla}}$ coincides with the standard divergence of tensor fields.

Proposition 4.1.1. *Let* (X, g, ω_X) *be an oriented smooth manifold of dimension* n. *The following map is a linear isomorphism:*

$$TX \otimes E \xrightarrow{\sim} \Lambda^{n-1}T^*X \otimes E \quad , \quad D \otimes e \mapsto i_D\omega_X \otimes e \ .$$

Moreover, if J *and* Π *are a vector field and a* $(n-1)$*-form corresponding via this isomorphism, then:*

$$\mathrm{d}_{\nabla'}\Pi = \omega_X \otimes \mathrm{div}_{\widetilde{\nabla}} J \ .$$

Proof: The isomorphism being clear, let us only check the differential property. On a neighbourhood of a point $x \in X$, let $D_1, \ldots, D_n$ be an orthonormal basis of TX and let $e_1, \ldots, e_m$ be a basis of E such that $(\mathrm{d}_\nabla D_j)_x = 0 = (\mathrm{d}_{\nabla'} e_i)_x$.

Let $J = J^{ij} D_i \otimes e_j$ be a vector field with values on E and let $\Pi = J^{ij} i_{D_i}\omega_X \otimes e_j$ be the corresponding form (we use summation over repeated indices).

Then:

$$\begin{aligned}
(\mathrm{div}_{\widetilde{\nabla}} J)_x &= c_1^1 \left(\mathrm{d}J^{ij} \otimes D_i \otimes e_j + J^{ij}(\mathrm{d}_\nabla D_i) \otimes e_j + J^{ij} D_i \otimes (\mathrm{d}_{\nabla'} e_j)\right)_x \\
&= c_1^1 \left(\mathrm{d}J^{ij} \otimes D_i \otimes e_j\right)_x = \left((D_i J^{ij}) e_j\right)_x \ . \\
(\mathrm{d}_{\nabla'}\Pi)_x &= \left((\mathrm{d}J^{ij} \wedge i_{D_i}\omega_X) \otimes e_j + J^{ij}(D_i^L \omega_X) \otimes e_j + J^{ij} i_{D_i}\omega_X \otimes (\mathrm{d}_{\nabla'} e_j)\right)_x \\
&= \left((i_{D_i}\mathrm{d}J^{ij})\omega_X \otimes e_j + J^{ij}(\mathrm{div} D_i)\omega_X \otimes e_j\right)_x = \left(\omega_X \otimes (D_i J^{ij}) e_j\right)_x \ .
\end{aligned}$$

□

Corollary 4.1.2. *The following map is a linear isomorphism:*

$$TX \otimes TX \xrightarrow{\sim} \Lambda^{n-1}T^*X \otimes TX \quad , \quad D \otimes D' \mapsto i_D\omega_X \otimes D' \ .$$

Moreover, if T *and* Π_{n-1} *are a 2-tensor and a* $(n-1)$*-form corresponding via this isomorphism, then:*

$$\mathrm{d}_\nabla \Pi_{n-1} = 0 \qquad \Leftrightarrow \qquad \mathrm{div}\, T = 0 \ .$$

Definition. Returning to the case $n = 4$, the energy-momentum tensor T of a fluid is the 2-contravariant tensor corresponding to the impulse form Π_3 via the isomorphism above.

Example. We have already showed that the impulse form of a dust with density ρ and velocity U is $\Pi_3 = \rho(i_U\omega_X)\otimes U$.

The corresponding energy-momentum tensor is:

$$T = \rho U \otimes U \ .$$

As $\operatorname{div} T = (\operatorname{div}(\rho U))U + \rho\nabla_U U$ and these two addends are orthogonal, the infinitesimal conservation of impulse $\operatorname{div} T = 0$ is equivalent to the following two conditions:

1. Infinitesimal Conservation of Mass: $\operatorname{div}(\rho U) = 0$.

2. Freely Falling Motion: $\nabla_U U = 0$ in the open set $\rho \neq 0$.

4.1.2 Dimensional analysis in Newtonian gravitation

Let us informally discuss the dimensional analysis in the Newtonian gravitation on $X = \mathbb{A}_4$.

When lengths and measures of mass and time are modified by some positive factors λ, μ, τ, if a magnitude, or a mathematical concept, is affected by the factor $\lambda^a\mu^b\tau^c$, then we say that the dimensions of this magnitude are $\mathsf{L}^a\mathsf{M}^b\mathsf{T}^c$. If the magnitude remains unchanged (i.e., $a = b = c = 0$), it is said to be independent of the units of scale.

Examples. The time metric $g = \mathrm{d}t^2$ in the Galilean spacetime has dimension T^2 and the space metric $\mathrm{d}x_1^2+\mathrm{d}x_2^2+\mathrm{d}x_3^2$, has dimension L^2. The hypervolume form $\omega_X = \mathrm{d}t\wedge\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3$ has dimensions $\mathsf{L}^3\mathsf{T}$, and we write

$$[\omega_X] = \mathsf{L}^3\mathsf{T} \ .$$

The unitary tangent vector U to a particle satisfies $[U] = \mathsf{T}^{-1}$, so that its impulse $I = mU$ has dimensions $\mathsf{T}^{-1}\mathsf{M}$. Consequently, the impulse form of a continuum of particles also satisfies $[\Pi_3] = \mathsf{T}^{-1}\mathsf{M}$.

Hence, the contravariant energy-momentum tensor T^2, defined by the formula $\Pi_3 = c_1^1(\omega_X \otimes T^2)$ has dimensions $[T^2] = \mathsf{M}\mathsf{T}^{-2}\mathsf{L}^{-3}$, and the corresponding covariant version:

$$[T_2] = \mathsf{M}\mathsf{T}^2\mathsf{L}^{-3} \ . \tag{4.1.2.1}$$

Law of Universal Gravitation

Assume that the force acting on two punctual masses m_1 and m_2, with distance r between them, is proportional to

$$\frac{m_1 m_2}{r^b}$$

for some positive real number $b \in \mathbb{R}^+$; that is, we assume that the force decreases with distance.

Fixing units of length, mass and time so that the universal gravitational constant becomes 1, it follows that $\frac{m_1 m_2}{r^b}$ has also dimensions of a force. Hence:

$$\mathsf{M}^2\mathsf{L}^{-b} = \mathsf{MLT}^{-2} \quad \Rightarrow \quad \mathsf{M} = \mathsf{L}^{1+b}\mathsf{T}^{-2} \, .$$

Introducing this relation into the dimensional analysis (4.1.2.1) of the mater tensor T_2, we observe that *a Newtonian gravitational force that decreases with distance corresponds to* $[T_2] = \mathsf{L}^w$ *with* $w = b - 2 > -2$*, whereas the inverse square force law,* $b = 2$*, corresponds to the condition of* T_2 *being independent of the units of scale (*$w = b - 2 = 0$*).*

4.2 Characterizations of the Einstein tensor

Let us now prove that the divergence-free condition, together with the independence of the unit of scale, characterize the Einstein tensor of a pseudo-Riemannian manifold.

In this Section, X will denote a manifold of arbitrary dimension n, and $Metrics$ will stand for the sheaf of pseudo-Riemannian metrics on X with a fixed signature.

Lemma 4.2.1. *If* $T\colon Metrics \longrightarrow 2\text{-}Tensors$ *is a natural tensor independent of the unit of scale, then it is an* $\mathbb{R}$*-linear combination of the following tensors:*

$$Ric \quad , \quad rg \, .$$

Proof: We apply Theorem 3.4.2. In the case $p = 2$ and $w = 0$, formula (3.4.0.5) reads:

$$2d_2 + \ldots + r\,d_r = 2 \, .$$

As the only solution is $d_2 = 1$, $d_3 = \ldots = d_r = 0$, the space of tensors under consideration is isomorphic to:

$$\mathrm{Hom}_{O_{g_x}}(N_2 \, , \, T_x^*X \otimes T_x^*X) = \mathrm{Hom}_{O_{g_x}}(N_2 \otimes T_xX \otimes T_xX \, , \, \mathbb{R}) \, .$$

The invariant theory of the orthogonal group (Theorem A.0.5) assures that the latter vector space is spanned by iterated contractions. Due to the symmetries of N_2, these generators reduce, up to signs, to the following two:

$$T^{ijijkk} \quad , \quad T^{ijikjk}$$

whose corresponding natural tensors are rg and Ric, respectively.

□

Theorem 4.2.2. *Let T: $Metrics \to$ 2-$Tensors$ be a natural tensor satisfying:*

1. *It is homogeneous of weight $w > -2$; that is, there exists a real number $w > -2$ such that, for any $\lambda \in \mathbb{R}^+$:*
$$T(\lambda^2 g) = \lambda^w T(g) \ .$$
2. *It is divergence-free.*

Then, there exist constants $\mu, \Lambda \in \mathbb{R}$ such that:

$$T = \begin{cases} \Lambda g & \text{if } w = 2. \\ \mu\left(Ric - \frac{r}{2} g\right) & \text{if } w = 0. \end{cases}$$

Proof: By Theorem 3.4.2, we have to study the non-negative integer solutions d_i to the equation:

$$2d_2 + \ldots + r\,d_r = 2 - w \ , \tag{4.2.0.2}$$

where $w > -2$. In particular, observe w has to be an integer number.

In case $w > 2$ or $w = 1$, this equation has no solutions, so there are no natural tensors with those weights.

If $w = 2$, there only exists the trivial solution $d_2 = \ldots = d_r = 0$. Moreover,

$$\mathrm{Hom}_{O_{g_x}}(\mathbb{R}, \otimes^2 T_x^* X) = (\otimes^2 T_x^* X)^{O_{g_x}} = \langle g_x \rangle \ ,$$

so the vector space of natural 2-tensors homogeneous of weight $w = 2$ is one-dimensional. Hence, it is generated by the metric g, which is divergence-free.

The case $w = 0$ follows from the previous Lemma and the identities:

$$\mathrm{div}(Ric) = \frac{1}{2}\mathrm{grad}\, r \quad , \quad \mathrm{div}(rg) = \mathrm{grad}\, r \ . \tag{4.2.0.3}$$

Finally, if $w = -1$ the only solution to (4.2.0.2) is $d_3 = 1$, $d_2 = d_4 = \ldots = d_r = 0$, that corresponds to the vector space of O_{g_x}-equivariant linear maps $N_3 \longrightarrow \otimes^2 T_x^* X$.

Nevertheless, there are no natural tensors with this weight, because there are no such maps:

$$\mathrm{Hom}_{O_{gx}}(N_3, \otimes^2 T_x^* X) \simeq \mathrm{Hom}_{O_{gx}}(N_3 \otimes T_x X \otimes T_x X, \mathbb{R}) = 0 ,$$

where the last equality easily follows because the total order (covariant plus contravariant) of the space of tensors $N_3 \otimes T_x X \otimes T_x X$ is odd (Theorem A.0.5).

$\square$

Remarks. This statement also holds, with analogous proof, in presence of an orientation and for dimension $n > 2$: if a divergence-free, natural operator

$$T : Metrics \times Orient \to 2\text{-}Tensors$$

is homogeneous of weight $w > -2$ (i.e., there exists $w > -2$ satisfying $T(\lambda^2 g, or) = \lambda^w T(g, or)$, for all $\lambda \in \mathbb{R}^+$), then there exist $\Lambda, \mu \in \mathbb{R}$ such that:

$$T = \begin{cases} \Lambda g & \text{if } w = 2 , \qquad \Lambda \in \mathbb{R} . \\ \mu\left(Ric - \frac{r}{2} g\right) & \text{if } w = 0 , \qquad \mu \in \mathbb{R} . \end{cases}$$

If X is of dimension 2, then the Einstein tensor is zero, but the homogeneity condition also allows, in the oriented case, the natural tensors $\omega_X, r\omega_X$. Among the linear combinations of $Ric = \frac{r}{2} g$, g, $r\omega_X$ and ω_X, only those involving g and ω_X are divergence-free.

As it has already been mentioned, this theorem, probably the most significant of the thesis, discards the possibility of another field equation, different from Einstein's equation - and hence an alternative law of universal gravitation, different from the inverse-square force law - on the grounds of the conservation of mass-energy, $\mathrm{div}\, T = 0$, and simple dimensional analysis.

Due to its importance, let us highlight the case $w = 0$, that provides a new characterization of the Einstein tensor:

Corollary 4.2.3. *If $T : Metrics \longrightarrow 2\text{-}Tensors$ is a natural tensor satisfying:*

1. *It is independent of the unit of scale.*
2. *It is divergence-free.*

then T is a constant multiple of the Einstein tensor:

$$T = \mu\left(Ric - \frac{r}{2} g\right) , \qquad \mu \in \mathbb{R} .$$

Remark. This Corollary is closely related to a result of Aldersley ([1]). This author considers a divergence-free, 2-contravariant tensor A^2 constructed from a metric g, whose coefficients A^{ij} depend on a finite number of derivatives of the coefficients of the metric. It is also assumed that these coefficients satisfy, with respect to a suitable system of coordinates, the following condition (that he calls *axiom of dimensional analysis*):

$$A^{ij}(g_{rs}, \lambda g_{rs,t_1}, \lambda^2 g_{rs,t_1 t_2}, \dots, \lambda^k g_{rs,t_1 \cdots t_k}) = \lambda^2 A^{ij}(g_{rs}, g_{rs,t_1}, \dots, g_{rs,t_1 \cdots t_k})$$

for all $\lambda > 0$. Then it is proved that A^2 coincides (up to a constant factor) with the contravariant Einstein tensor T^2.

Although the above axiom is not intrinsic, it is not difficult to show that Aldersley's axiom for A^2 is equivalent, in the case of a natural tensor, to the condition of A^2 having weight -4 or, in other words, of A_2 having weight 0, where A_2 is the 2-covariant tensor metrically equivalent to A^2.

4.2.1 Classical results

Let us consider a second-order, natural tensor $T\colon Metrics \longrightarrow 2\text{-}Tensors$; i.e., a morphism of natural bundles $T\colon J^2M \longrightarrow \otimes^2 TX$, which is linear in the second derivatives of the metric.

The following classical theorem, with the additional assumption of T being symmetric, is generally attributed to H. Vermeil:

Theorem 4.2.4. *Let $T\colon Metrics \to 2\text{-}Tensors$ be a second-order, natural tensor satisfying:*

- *It is linear on the second derivatives of the metric.*
- *It is divergence-free.*

Then there exist universal constants $\mu, \Lambda \in \mathbb{R}$ such that:

$$T = \mu\left(Ric - \frac{r}{2} g\right) + \Lambda g\ .$$

Proof: Fix a metric g_x at a point $x \in X$. By Corollary 3.3.9, a second-order, natural tensor T corresponds to a smooth O_{g_x}-equivariant map:

$$\mathfrak{t}\colon N_2 \longrightarrow \otimes^2 T_x^* X\ .$$

The condition of T being linear on the second derivatives of the metric amounts to saying that $\mathfrak{t}$ is an affine map. Then, the tangent linear map is the same at any point and is an O_{g_x}-equivariant linear map $\mathfrak{t}_* : N_2 \longrightarrow \otimes^2 T_x^* X$.

We have already showed that the vector space of these linear maps has dimension two (see the proof of Lemma 4.2.1), and the corresponding space of natural tensors is $\langle Ric, rg\rangle$. On the other hand, the O_{g_x}-equivariant map $\mathfrak{t} - \mathfrak{t}_*$ is constant; hence it corresponds to an element of $\left(\otimes^2 T_x^* X\right)^{O_{g_x}} = \langle g \rangle$.

That is, all the second-order, natural tensors T that are linear in the second derivatives of the metric are the linear combinations of Ric, rg, and g.

The identities (4.2.0.3) allow to conclude the proof.

□

In dimension 4, D. Lovelock showed that the linearity assumption can be removed:

Theorem 4.2.5 (Lovelock, [31])**.** *Let $T : Metrics \to 2\text{-}Tensors$ be a natural tensor satisfying:*

- *It is second-order.*
- *It is divergence-free.*

If X is of dimension 4, then there exist $\mu, \Lambda \in \mathbb{R}$ such that:

$$T = \mu\left(Ric - \frac{r}{2} g\right) + \Lambda g \ .$$

We detail a new proof of this result in Chapter VI.

4.2.2 Characterization on a Weyl geometry

Definition. A Weyl spacetime $(X, \nabla, <g>)$ consists of a smooth manifold X of dimension $1+n$, endowed with a symmetric linear connection ∇ and a conformal lorentzian structure $<g>$, both of which are compatible in the sense that:

$$\nabla g = \alpha \otimes g$$

for some 1-form α.

This 1-form α depends on the chosen representative g of the conformal structure $<g>$. As the 1-forms associated to different representatives differ on an exact form, the 2-form $\omega := \mathrm{d}\alpha$ is intrinsic, and is called the *Weyl form*.

It holds:

$$\mathrm{d}_\nabla^2 g = \omega \otimes g$$

so the Weyl 2-form ω is the curvature 2-form of ∇, understood as a connection on the line bundle $<g>$. As a consequence:

Proposition 4.2.6. *The vanishing of ω amounts to the local existence of a representative g of the conformal structure such that $\nabla g = 0$.*

In other words, the vanishing of ω amounts, locally, to ∇ being the Levi-Civita connection of some representative g of the conformal structure (this representative is unique, up to a constant factor).

Nevertheless, even in the case $\omega = 0$, the Weyl geometry may not be globally Lorentzian, the obstruction being the cohomology class $[\alpha] \in H^1(X,\mathbb{R})$. Hence, topological assumptions (v.gr., if X is simply connected) may guarantee that, if $\omega = 0$, then there exists a *global* representative g of the conformal structure such that $\nabla g = 0$.

For a fixed representative g of the conformal structure, the compatibility with ∇ reads:

$$D_0(g(D_1,D_2)) = g(\nabla_{D_0}D_1,D_2) + g(D_1,\nabla_{D_0}D_2) + \alpha(D_0)g(D_1,D_2)\ .$$

Analogously to construction of the Levi-Civita connection of a pseudo-Riemannian metric, it can be proved:

Fundamental Lemma. *For any given Lorentzian metric g and any 1-form α, there exists a unique symmetric linear connection ∇ such that $\nabla g = \alpha \otimes g$.*

Therefore, any pair (g,α) of a Lorentz metric and a 1-form α on X, defines a Weyl spacetime $(X,\nabla,<g>)$.

Proposition 4.2.7. *Let $(X,\nabla,<g>)$ be a Weyl spacetime, and let us fix a representative g of the conformal structure.*

The difference between the connection ∇ and the Levi-Civita connection ∇_g of g is the following $(2,1)$-tensor:

$$\gamma(D_1,D_2) := (\nabla_{D_1} - (\nabla_g)_{D_1})D_2 = \frac{1}{2}\left(g(D_1,D_2)A - \alpha(D_1)D_2 - \alpha(D_2)D_1\right)\ ,$$

where A is the vector field metrically equivalent to the 1-form α.

Proof: The skew-symmetric part (in the covariant indices) of the difference γ between two connections equals the difference between their torsion tensors. As both connections are symmetric, γ is a symmetric tensor.

Subtracting the following equations:

$$\begin{aligned} D_0(g(D_1,D_2)) &= g(\nabla_{D_0}D_1,D_2)+g(D_1,\nabla_{D_0}D_2)+\alpha(D_0)g(D_1,D_2)\ , \\ D_0(g(D_1,D_2)) &= g((\nabla_g)_{D_0}D_1,D_2)+g(D_1,(\nabla_g)_{D_0}D_2)\ , \end{aligned}$$

we obtain:

$$0 = g(\gamma(D_0,D_1),D_2)+g(D_1,\gamma(D_0,D_2))+\alpha(D_0)g(D_1,D_2)\ .$$

The thesis follows by adding three times this equation, with adequate sign and with a cyclic permutation of its indices. □

Computations

Let $(X,\nabla,<g>)$ a Weyl spacetime and $p \in X$ be a point.

Consider a chart $x_0,\dots,x_n$ centred at p and such that the Christoffel symbols of ∇ vanish at p:

$$\Gamma^k_{ij}(p) = 0 \qquad , \qquad (\nabla \mathrm{d}x_k)_p = 0\ ,$$

and such that the metric:

$$g_p := \sum_{i=0}^{n} \delta_i \mathrm{d}x_i \otimes \mathrm{d}x_i$$

where $\delta_0 = 1$, $\delta_i = -1$, $i = 1,\dots,n$, represents the conformal structure at p.

Let us extend g_p by parallel transport along the radial lines, so that we obtain a metric g in a neighbourhood of p satisfying:

$$\nabla_H g = 0\ ,$$

where $H = \sum_{i=0}^n x_i \partial_{x_i}$ is the field of the homotheties on those coordinates.

Let $\alpha = \sum_{i=0}^n \alpha_i \mathrm{d}x_i$ be the 1-form associated to g. As

$$0 = \nabla_{(\sum_{i=0}^n x_i\partial_{x_i})} g = \alpha\left(\sum_{i=0}^{n} x_i \partial_{x_i}\right) \otimes g\ ,$$

it follows

$$0 = \alpha\left(\sum_{i=0}^{n} x^i \partial_{x^i}\right) = \sum_{i=0}^{n} \alpha_i x^i\ . \tag{4.2.2.1}$$

Lemma 4.2.8. *With the previous notations, it holds:*

$$\alpha_p = 0 \quad , \quad (\nabla\alpha)_p = \frac{1}{2}(\mathrm{d}\alpha)_p = \frac{1}{2}\omega_p \quad , \quad \frac{\partial g_{ij}}{\partial x^k}(p) = 0 \ .$$

Proof: Differentiating in (4.2.2.1) and taking value at p implies $\alpha_i(p) = 0$, so that $\alpha_p = 0$.

Differentiating twice and taking value at the origin, results $\frac{\partial\alpha_i}{\partial x^j}(p) + \frac{\partial\alpha_j}{\partial x^i}(p) = 0$, so that:

$$\begin{aligned}
(\nabla\alpha)_p &= \left(\sum_{i=0}^{n} \mathrm{d}\alpha_i \otimes \mathrm{d}x^i\right)_p = \sum_{i,j} \frac{\partial\alpha_i}{\partial x^j}(p)\,\mathrm{d}x^j \otimes \mathrm{d}x^i \\
&= \frac{1}{2}\sum_{i,j}\left(\frac{\partial\alpha_i}{\partial x^j}(p) + \frac{\partial\alpha_i}{\partial x^j}(p)\right)\mathrm{d}x^j \otimes \mathrm{d}x^i + \frac{1}{2}\sum_{i,j}\left(\frac{\partial\alpha_i}{\partial x^j}(p) - \frac{\partial\alpha_i}{\partial x^j}(p)\right)\mathrm{d}x^i \otimes \mathrm{d}x^j \\
&= 0 + \frac{1}{2}(\mathrm{d}\alpha)_p \ .
\end{aligned}$$

Finally, the metric g is parallel along the lines $\{x^0 = \lambda^0 t, \ldots, x^n = \lambda^n t\}$, so that:

$$0 = \left(\nabla_{(\sum_{k=0}^{n}\lambda^k \partial_{x^k})} g\right)_p = \sum_{i,j}\left(\sum_{k=0}^{n} \lambda^k \frac{\partial g_{ij}}{\partial x^k}\right)(p)\,\mathrm{d}x^i \otimes \mathrm{d}x^j \ ,$$

and, as the λ^i are arbitrary, it follows $\frac{\partial g_{ij}}{\partial x^k}(p) = 0$. □

The conditions $\frac{\partial g_{ij}}{\partial x^k}(p) = 0$ imply that the Christoffel symbols of the Levi-Civita connection ∇_g of g satisfy $\overline{\Gamma}^k_{ij}(p) = 0$, so that $(\nabla_g)_p = \nabla_p$.

Lemma 4.2.9. *With the previous notations, it holds:*

$$(Ric_\nabla)_p = (Ric_g)_p - \frac{1+n}{4}\,\omega_p$$

where Ric_∇ and Ric_g denote the Ricci tensors of ∇ and g, respectively.

Proof: Let R_{jh} and $\overline{R}_{jh}$ be the coefficients at p of the tensors Ric_∇ and Ric_g, respectively. Let us use the following standard formulae:

$$\begin{aligned}
R_{jh} &= \sum_{i=0}^{n}\left(\Gamma^i_{jh,i} - \Gamma^i_{ih,j}\right) \\
\overline{R}_{jh} &= \sum_{i=0}^{n}\left(\overline{\Gamma}^i_{jh,i} - \overline{\Gamma}^i_{ih,j}\right)
\end{aligned}$$

where the index after the colon denotes partial derivative at p.

On the other hand,

$$\Gamma^i_{jh} - \overline{\Gamma}^i_{jh} = \gamma(\partial_{x_j}, \partial_{x_h})^i \overset{4.2.7}{=} \frac{1}{2}\left(g_{jh}A^i - \alpha_j \delta_{hi} - \alpha_h \delta_{ji}\right) .$$

Therefore,

$$\begin{aligned} R_{jh} - \overline{R}_{jh} &= \sum_{i=0}^{n}\left(\Gamma^i_{jh,i} - \Gamma^i_{ih,j}\right) - \sum_{i=0}^{n}\left(\overline{\Gamma}^i_{jh,i} - \overline{\Gamma}^i_{ih,j}\right) \\ &= \sum_{i=0}^{n}\frac{1}{2}\left(g_{jh}A^i - \alpha_j\delta_{hi} - \alpha_h\delta_{ji}\right)_{,i} - \sum_{i=0}^{n}\frac{1}{2}\left(g_{ih}A^i - \alpha_i\delta_{hi} - \alpha_h\delta_{ii}\right)_{,j} \\ &= \sum_{i=0}^{n}\frac{1}{2}\left(g_{jh}(p)A^i{}_{,i} - \alpha_{j,i}\delta_{hi} - \alpha_{h,i}\delta_{ji}\right) - \sum_{i=0}^{n}\frac{1}{2}\left(g_{ih}(p)A^i{}_{,j} - \alpha_{i,j}\delta_{hi} - \alpha_{h,j}\delta_{ii}\right) \\ &= \frac{1}{2}\left(g_{jh}(p)\sum_{i=0}^{n}A^i{}_{,i} - \alpha_{j,h} - \alpha_{h,j}\right) - \frac{1}{2}\left(g_{hh}(p)A^h{}_{,j} - \alpha_{h,j} - (1+n)\alpha_{h,j}\right) . \end{aligned}$$

As $(\nabla\alpha)_p$ is skew-symmetric, we have $A^i{}_{,i} = \delta_i\alpha_{i,i} = 0$, so the previous computation equals:

$$R_{jh} - \overline{R}_{jh} = \frac{1}{2}(-\alpha_{j,h} - \alpha_{h,j}) + \frac{1+n}{2}\alpha_{h,j} = -\frac{1+n}{4}(\omega_p)_{jh} ,$$

the last equality because $\alpha_{j,h} + \alpha_{h,j} = 0$ (recall $(\nabla\alpha)_p$ is skew-symmetric), and $\alpha_{h,j}$ is the coefficient jh of $(\nabla\alpha)_p = \frac{1}{2}\omega_p$. □

Corollary 4.2.10. *The skew-symmetric component of Ric_∇ is $-\frac{1+n}{4}\omega$.*

Let R^k_{ijh} denote the coefficients at p of the curvature tensor R, let $R^k_{ijh,s}$ be its partial derivatives at p, and let ω_{ij} be the coefficients of ω at p.

The proof of the following proposition is routine:

Proposition 4.2.11. *The curvature tensor R possesses the the following symmetries:*

- *It is skew-symmetric in the first two indices: $R^k_{ijh} = -R^k_{jih}$.*
- *The last two indices satisfy: $\delta_k R^k_{ijh} = -\delta_h R^h_{ijk}$ for $k \neq h$, and $R^k_{ijk} = -\frac{1}{2}\omega_{ij}$.*
- *Bianchi linear identity: $R^k_{ijh} + R^k_{hij} + R^k_{jhi} = 0$.*
- *Bianchi differential identity: $R^k_{ijh,s} + R^k_{sih,j} + R^k_{jsh,i} = 0$.*

Tensors associated to a Weyl spacetime

If $(X, \nabla, <g>)$ is a Weyl spacetime, consider the following covariant 2-tensors: Ric_∇, ω and rg, where r is the total contraction of Ric_∇ respect to any metric g of the conformal structure (the value of r depends on the chosen metric, but the tensor rg is intrinsic). Our purpose is to show that any 2-tensor naturally associated to the Weyl spacetime is a linear combination of these three tensors.

Weyl geometries on a manifolds do not correspond with the sections of any bundle, so the notion of naturalness has to be redefined in this setting.

On a smooth manifold X, let $Weyl$ and $1\text{-}Forms$ denote the sheaf of Weyl structures and 1-forms on X, respectively. The fundamental lemma on page 58 allows to define a morphism of sheaves:

$$Metrics \times 1\text{-}Forms \xrightarrow{\varphi} Weyl \quad , \quad (g, \alpha) \longmapsto (\nabla, <g>)$$

Observe this map satisfies the homogeneity condition $\varphi(\lambda^2 g, \alpha) = \varphi(g, \alpha)$, for all $\lambda \in \mathbb{R}^+$.

Definition. A morphism of sheaves $T\colon Weyl \longrightarrow 2\text{-}Tensors$ is natural if the composition

$$Metrics \times 1\text{-}Forms \xrightarrow{\varphi} Weyl \xrightarrow{T} 2\text{-}Tensors$$

is a natural morphism.

The composition $T \circ \varphi$ fulfils the homogeneity condition

$$(T \circ \varphi)(\lambda^2 g, \alpha) = (T \circ \varphi)(g, \alpha) .$$

Lemma 4.2.12. *Any natural tensor $T'\colon Metrics \times 1\text{-}Forms \longrightarrow 2\text{-}Tensors$ satisfying the homogeneity condition $T'(\lambda^2 g, \alpha) = T'(g, \alpha)$, for all $\lambda \in \mathbb{R}^+$, is a linear combination of the following seven tensors:*

$$Ricci_g \quad , \quad r_g g \quad , \quad \alpha \otimes \alpha \quad , \quad |\alpha|^2 g$$

$$\nabla_g \alpha \quad , \quad \mathrm{tr}(\nabla_g \alpha) g \quad , \quad \mathrm{d}\alpha .$$

Proof: By Theorem 3.4.2, the space of tensors under consideration is isomorphic to the vector space of O_{g_x}-equivariant linear maps:

$$S^{d_2} N_2 \otimes \ldots \otimes S^{d_r} N_r \otimes S^{\bar{d}_0} \Lambda_0 \otimes \ldots \otimes S^{\bar{d}_s} \Lambda_s \longrightarrow \otimes^2 T_x^* X$$

where $\Lambda_m := T_x^*X \otimes S^m T_x^*X$, $m = 0, \ldots, s$, and the coefficients $d_i, \overline{d}_j \in \mathbb{N}$ satisfy the equation:

$$2d_2 + \ldots + rd_r + \overline{d}_0 + 2\overline{d}_1 + \ldots + (s+1)\overline{d}_s = 2 \; .$$

If some $\overline{d}_i$ is non zero, then there are only two possibilities:

• $\overline{d}_0 = 2$, $\overline{d}_1 = \ldots = \overline{d}_s = d_j = 0$. In this case, we are reduced to compute O_{g_x}-equivariant linear maps:

$$S^2(T^*X) \otimes T_xX \otimes T_xX \longrightarrow \mathbb{R} \; .$$

Those linear maps are linear combinations of iterated contractions; due to symmetries, any such an iterated contraction is a linear combination of:

$$T^{aabb} \qquad , \qquad T^{abab} \; .$$

These contractions, in turn, correspond, respectively, with the tensors:

$$|\alpha|^2 g \qquad , \qquad \alpha \otimes \alpha \; .$$

• $\overline{d}_1 = 1$, $\overline{d}_0 = \overline{d}_2 \ldots = \overline{d}_s = d_j = 0$: In this case, the space of O_{g_x}-equivariant linear maps:

$$T_x^*X \otimes T_x^*X \otimes T_xX \otimes T_xX \longrightarrow \mathbb{R}$$

has dimension 3, for it is generated by the total contractions:

$$T^{aabb} \qquad , \qquad T^{abab} \qquad , \qquad T^{abba} \; .$$

Therefore, the corresponding space of natural tensors is generated by:

$$\mathrm{tr}(\nabla_g \alpha) g \qquad , \qquad \nabla_g \alpha \qquad , \qquad \left(\nabla_g \alpha\right)^t \; ,$$

where $\left(\nabla_g \alpha\right)^t (D_1, D_2) := (\nabla_g \alpha)(D_2, D_2)$.

Therefore, this vector space is also generated by:

$$\mathrm{tr}(\nabla_g \alpha) g \qquad , \qquad \nabla_g \alpha \qquad , \qquad \mathrm{d}\alpha \; .$$

Finally, if the $\overline{d}_i$ are all zero, then the tensor T does not depend on α and therefore is a linear combination of Ric_g and $r_g g$ (Lemma 4.2.1). $\square$

Theorem 4.2.13. *Any natural tensor $T\colon Weyl \longrightarrow 2\text{-}Tensors$ is a linear combination of the following three tensors*

$$Ric_\nabla \quad , \qquad rg \quad , \qquad \omega \, .$$

Proof: By the previous lemma, the composition $T' = T \circ \varphi$ is a linear combination of seven tensors. Given a Weyl structure $(\nabla, <g>)$ and a point $p \in X$, let us consider a pair (g, α) as in the precedent subsection of computations.

By Lemma 4.2.8, it follows

$$\alpha_p = 0 \quad , \qquad (\nabla_g \alpha)_p = (\nabla \alpha)_p = \frac{1}{2}(\mathrm{d}\alpha)_p = \frac{1}{2}\omega_p \, .$$

Consequently, the tensors $(\alpha \otimes \alpha)_p$, $(|\alpha|^2 g)_p$ and $((\mathrm{div}_g \alpha) g)_p$ are all zero.

On the other hand, by Lemma 4.2.9, the tensor $(Ric_g)_p$ is a linear combination of $(Ric_\nabla)_p$ and ω_p.

That is, the tensor $T(\nabla, <g>)_p = T'(g, \alpha)_p$, that in principle is a linear combination of the seven tensors enumerated in Lemma 4.2.12, is indeed a linear combination of $(Ric_\nabla)_p$, $(rg)_p$ and ω_p.

$\square$

Definition. A 2-tensor T on a Weyl spacetime $(X, \nabla, <g>)$ is divergence-free if the contraction of the first two indices of ∇T (with respect to any metric of the conformal structure) is zero.

Given a point p on a Weyl spacetime $(X, \nabla, <g>)$, let us consider a pair (g, α), chosen as in the subsection of computations. Given a 2-tensor T, let us call *divergence* of T to the 1-form obtained by contracting the first two indices of ∇T with respect to the metric g. This notion is not intrinsic.

Lemma 4.2.14. *With the previous conventions, the divergence of the tensor Ric_∇ at the point p is*

$$\mathrm{div}(Ric_\nabla) = \frac{1}{2}\mathrm{div}(rg) - \frac{n}{4}\,\mathrm{div}\,\omega \, .$$

Proof: Let us compute the divergence of Ric at p. Using the Bianchi linear identity, it follows

$$
\begin{aligned}
(\mathrm{div} Ric)^j &= \sum_i \delta_i R_{ij,i} = \sum_{ik} \delta_i R^k_{kij,i} = -\sum_{ik} \delta_i R^k_{jki,i} - \sum_{ik} \delta_i R^k_{ijk,i} \\
&\overset{4.2.11}{=} -\sum_{ik} \delta_i R^k_{jki,i} + \frac{1}{2}\sum_i \delta_i \omega_{ij,i} = -\sum_{ik} \delta_i R^k_{jki,i} + \frac{1}{2}(\mathrm{div}\,\omega)^j \ .
\end{aligned}
$$

Applying the Bianchi differential identity, one of the last addends is equal to:

$$
\begin{aligned}
-\sum_{ik} \delta_i R^k_{jki,i} &= \sum_{ik} \delta_i R^k_{iji,k} + \sum_{ik} \delta_i R^k_{kii,j} = \sum_{ik} \delta_i R^k_{iji,k} + r_{,j} \\
&= \sum_{ik} \delta_i R^k_{iji,k} + (\mathrm{div}(rg))^j \ .
\end{aligned}
$$

Applying the symmetry of the last 2 indices of R (item *2* in Proposition 4.2.11),

$$
\sum_{ik} \delta_i R^k_{iji,k} = -\sum_{ik} \delta_k R^i_{ijk,k} = -\sum_k \delta_k R_{jk,k} = -(\mathrm{div} S)^j
$$

where S is the symmetric part of Ric_∇. According to Corollary 4.2.10, the skew-symmetric component of Ric_∇ is $-\frac{1+n}{4}\omega$, so that $S = Ric_\nabla + \frac{1+n}{2}\omega$.

Summing up,

$$
\mathrm{div}(Ric_\nabla) = \frac{1}{2}\mathrm{div}\,\omega + \mathrm{div}(rg) - \mathrm{div}\left(Ric_\nabla + \frac{1+n}{2}\omega\right) ,
$$

and the thesis follows.

□

As a consequence of this Lemma and Theorem 4.2.13, it follows:

Theorem 4.2.15. *Up to a constant factor, the only natural 2-tensor* $G\colon Weyl \longrightarrow 2\text{-}Tensors$ *that is divergence-free is:*

$$
G := Ric_\nabla - \frac{r}{2} g + \frac{n}{4}\omega \qquad\qquad (\dim X = 1+n) \ .
$$

Observe this tensor G is *not* symmetric; its skew-symmetric component is

$$
-\frac{1+n}{4}\omega + \frac{n}{4}\omega = -\frac{1}{4}\omega \ .
$$

Chapter 5

Characterization of the electromagnetic energy tensor

Let X be a smooth manifold of dimension $1+n$ and let g be a Lorentzian metric on it, of signature $(+,-,\dots,-)$. An electromagnetic field on X is represented by a differential 2-form F, and its electromagnetic energy tensor T is a 2-covariant tensor defined in a local chart by the formula:

$$T_{ab} := -\left(F_a{}^i F_{bi} - \frac{1}{4}F^{ij}F_{ij}g_{ab}\right) . \tag{5.0.2.1}$$

As it has been explained in Chapter 4, the matter content of a relativistic spacetime X is represented by a 2-covariant tensor T_{m}, called the matter energy-momentum tensor, and, in absence of electric charges, the mass-energy and impulse conservation laws are encoded in the equation:

$$\operatorname{div} T_{\mathrm{m}} = 0 .$$

Nevertheless, when dealing with charged matter, the Lorentz force law imposes:

$$\operatorname{div} T_{\mathrm{m}} = i_J F = i_{\partial F} F$$

where J is the charge-current vector field and $\partial F = J^*$ because of the second Maxwell equation (see our notations in page 68).

Therefore, in order to fulfil the aforementioned conservation laws, it is necessary to assume that, apart from the energy-momentum tensor T_{m} of the matter distribution, there also exists some kind of energy associated to the electromagnetic field itself, represented by some

2-covariant tensor T_{elm}, such that:

$$\operatorname{div}(T_{\mathrm{m}} + T_{\mathrm{elm}}) = 0 \ . \tag{5.0.2.2}$$

Of course, this equality implies $\operatorname{div} T_{\mathrm{elm}} = -i_{\partial F} F$.

As regards to dimensional analysis, it is easy to check that a change of the time unit $\overline{g} = \lambda^2 g$ implies a modification of the type $\overline{F} = \lambda F$ in the mathematical representation of the electromagnetic field. In Chapter 4 we explained that the matter tensor remains invariable, $\overline{T}_{\mathrm{m}} = T_{\mathrm{m}}$, so that for equation (5.0.2.2) to make sense it is necessary that the electromagnetic energy tensor also stands invariable: $\overline{T}_{\mathrm{elm}} = T_{\mathrm{elm}}$.

Finally, it is sensible to assume that the electromagnetic energy is null wherever the field is null.

Summing up, these are three properties that have to be satisfied by any physically reasonable definition of electromagnetic energy. Our main result, Theorem 5.2.4, proves that these conditions uniquely characterize the energy tensor (5.0.2.1) of a 2-form.

Finally, the existence of an energy tensor associated to a differential k-form, with k arbitrary, suggests the question of a possible physical interpretation for it. In the last two Sections, we consider a generalized theory of electromagnetism for charged p-branes, introduced by Henneaux and Teitelboim ([15]), where the electromagnetic field F is a differential form of order $2+p$. We extend this theory up to the point of including fluids of charged p-branes; the corresponding Maxwell-Einstein equations require an electromagnetic energy tensor, which turns out to be the energy tensor associated to the form F.

The original results of this Chapter have been published in [38].

5.1 Energy tensor of a differential form

Throughout this Chapter, let (X, g, ω_X) be an oriented and time-oriented pseudo-Riemannian manifold of dimension $1+n$, whose metric has signature $(+,-,\overset{n}{\ldots},-)$. We adopt this signature for consistency with the last two Sections, although the results of the first two are valid on pseudo-Riemannian manifolds or arbitrary signature.

Given a q-vector $D_1 \wedge \ldots \wedge D_q$ and a differential k-form ω, with $q \leq k$, we write:

$$i_{D_1 \wedge \ldots \wedge D_q} \omega := i_{D_q} \ldots i_{D_1} \omega = \omega(D_1, \ldots, D_q, \_, \ldots, \_) \ .$$

Analogously, if ω is a k-form and $\bar{\omega}$ is a q-form, with $q \leq k$, we write:

$$i_{\bar{\omega}}\omega := i_{\bar{\omega}^*}\omega$$

where $\bar{\omega}^*$ is the q-vector metrically equivalent to $\bar{\omega}$.

With these notations, the metric induced on the bundle of k-forms is:

$$\langle \omega, \bar{\omega} \rangle := i_\omega \bar{\omega} = \frac{1}{k!}\omega^{j_1 \dots j_k}\bar{\omega}_{j_1 \dots j_k} \ .$$

The Hodge star is the linear isomorphism $*\colon \Lambda^k T^*X \to \Lambda^{1+n-k}T^*X$ defined as

$$*\omega := i_\omega \omega_X \ ,$$

and, with these conventions, it holds $**\omega = (-1)^{(k+1)n}\omega$.

The codifferential $\partial\colon \Lambda^k T^*X \rightsquigarrow \Lambda^{k-1}T^*X$ is the following differential operator:

$$\partial := (-1)^{(1+n)k} * \mathrm{d} * \ , \qquad \text{or, equivalently,} \qquad * \partial := (-1)^k \mathrm{d} * \ .$$

In a local chart:

$$(\partial\omega)_{i_1 \dots i_{k-1}} = -\nabla^a \omega_{a i_1 \dots i_{k-1}} \ .$$

Definition and main properties

Let ω be a differential k-form on X.

Definition. Let U be an observer at a point x (that is, U is a unitary timelike vector oriented to the future). Let us consider an orthonormal frame $(D_0 = U, D_1, \dots, D_n)$ of T_xX and the corresponding dual base $(\theta_0 = U^*, \theta_1, \dots, \theta_n)$.

In terms of this basis, the k-form ω decomposes as a multiple of θ_0, called the *electric part* E_U, and other terms without θ_0, called the *magnetic part* B_U:

$$\omega = E_U + B_U = (\text{terms with } \theta_0) + (\text{terms without } \theta_0) \ .$$

In other words:

$$E_U := U^* \wedge i_U \omega \qquad , \qquad B_U := i_U(U^* \wedge \omega) \ ,$$

so these k-forms E_U, B_U depend on the observer U but not on the chosen basis.

Moreover, as E_U and B_U are orthogonal:

$$\langle \omega, \omega \rangle \;=\; \langle E_U, E_U \rangle + \langle B_U, B_U \rangle \;.$$

These two addends have definite signs, that we modify to make them positive:

$$\begin{aligned} |E_U|^2 := (-1)^{k-1} \langle E_U, E_U \rangle \;&=\; (-1)^{k-1} \langle i_U \omega, i_U \omega \rangle \\ |B_U|^2 := (-1)^{k} \quad \langle B_U, B_U \rangle \;&=\; (-1)^{k} \quad \langle U^* \wedge \omega, U^* \wedge \omega \rangle \;. \end{aligned}$$

Hence,

$$\langle \omega, \omega \rangle \;=\; (-1)^{k-1} \left(|E_U|^2 - |B_U|^2 \right)$$

and the right hand side of this equation does not depend on the observer.

Definition. The energy of a differential k-form ω with respect to an observer U is the smooth function:

$$e(U) := \frac{1}{2} \left(|E_U|^2 + |B_U|^2 \right) \;.$$

Unfolding the definitions, we see the energy is quadratic on U:

$$\begin{aligned} e(U) &= \frac{1}{2} \left(|E_U|^2 + |B_U|^2 \right) = \frac{1}{2} \left(|E_U|^2 + (-1)^k \langle \omega, \omega \rangle + |E_U|^2 \right) \\ &= (-1)^{k-1} \left(\langle i_U \omega, i_U \omega \rangle - \frac{1}{2} \langle \omega, \omega \rangle \langle U, U \rangle \right) \;, \end{aligned}$$

so we are led to consider the corresponding symmetric tensor:

Definition. The energy tensor of a differential k-form ω is the 2-covariant symmetric tensor T defined as:

$$(-1)^{k-1} T(D_1, D_2) \;:=\; \langle i_{D_1} \omega, i_{D_2} \omega \rangle - \frac{1}{2} \langle \omega, \omega \rangle \, g(D_1, D_2) \;.$$

This definition is made so that $T(U,U) = e(U)$ for every observer U.

Remark. These energy tensors are a particular case of the *superenergy tensors* introduced by Senovilla ([43]): the superenergy tensor associated to a differential k-form is precisely the energy tensor defined above.

Nevertheless, the superenergy construction can be applied to any kind of tensors, not necessarily skew-symmetric, and the result obtained may have more than 2 indices, depending on the symmetries of the original tensor.

Next, we prove the main property of the energy tensors ([43], Thm. 4.1), although we will not use it in this memory:

Dominant Energy Condition: *For any pair U_1, U_2 of observers (unitary timelike vector fields oriented to the future), the energy tensor satisfies:*

$$T(U_1, U_2) \geq 0 \ .$$

Proof: Let U_1, U_2 be unitary timelike vector fields oriented to the future and let $D_0, D_1, \dots, D_n$ be an orthonormal basis such that $U_1 = D_0$ and $U_2 = D_0 + vD_1$, with $0 \leq v < 1$.

Let $\theta_0, \dots, \theta_n$ be the dual basis and let us write:

$$\omega = \theta_0 \wedge \omega_\alpha + \theta_1 \wedge \omega_\beta + \theta_0 \wedge \theta_1 \wedge \omega_\gamma + \omega_\delta$$

where $\omega_\alpha, \omega_\beta, \omega_\gamma, \omega_\delta$ are not multiples neither of θ_0 nor of θ_1.

Then,

$$\begin{aligned}
(-1)^{k-1} T(U_1, U_2) &= (-1)^{k-1} \Big(T(D_0, D_0) + v\, T(D_0, D_1) \Big) \\
&= \langle i_{D_0}\omega, i_{D_0}\omega \rangle - \frac{1}{2}\langle \omega, \omega \rangle + v \langle i_{D_0}\omega, i_{D_1}\omega \rangle \\
&= \langle \omega_\alpha, \omega_\alpha \rangle - \langle \omega_\gamma, \omega_\gamma \rangle - \frac{1}{2}\Big(\langle \omega_\alpha, \omega_\alpha \rangle - \langle \omega_\beta, \omega_\beta \rangle - \langle \omega_\gamma, \omega_\gamma \rangle + \langle \omega_\delta, \omega_\delta \rangle \Big) \\
&\quad + v \,\langle \omega_\alpha + \theta_1 \wedge \omega_\gamma, \omega_\beta - \theta_0 \wedge \omega_\gamma \rangle \\
&= \frac{1}{2}\Big(\langle \omega_\alpha, \omega_\alpha \rangle + \langle \omega_\beta, \omega_\beta \rangle - \langle \omega_\gamma, \omega_\gamma \rangle - \langle \omega_\delta, \omega_\delta \rangle \Big) + v \langle \omega_\alpha, \omega_\beta \rangle \\
&= \frac{1}{2}\Big(\langle \omega_\alpha + v\omega_\beta, \omega_\alpha + v\omega_\beta \rangle + (1 - v^2)\langle \omega_\beta, \omega_\beta \rangle - \langle \omega_\gamma, \omega_\gamma \rangle - \langle \omega_\delta, \omega_\delta \rangle \Big) \ .
\end{aligned}$$

That is to say, $2\,T(U_1, U_2)$ equals:

$$\begin{aligned}
(-1)^{k-1} \langle \omega_\alpha + v\omega_\beta, \omega_\alpha + v\omega_\beta \rangle + (-1)^{k-1}(1 - v^2)\langle \omega_\beta, \omega_\beta \rangle \\
+ (-1)^{k-2} \langle \omega_\gamma, \omega_\gamma \rangle + (-1)^k \langle \omega_\delta, \omega_\delta \rangle
\end{aligned}$$

which is a strictly positive quantity, because $\langle \omega_\alpha, \omega_\alpha \rangle = \langle \omega_\beta, \omega_\beta \rangle = (-1)^{k-1}$, $\langle \omega_\gamma, \omega_\gamma \rangle = (-1)^{k-2}$ and $\langle \omega_\delta, \omega_\delta \rangle = (-1)^k$. $\square$

For any observer U, the Hodge star maps the electric and magnetic parts of ω into the magnetic and electric parts (up to signs) of $*\omega$:

$$*E_U(\omega) = \pm B_U(*\omega) \qquad , \qquad *B_U(\omega) = \pm E_U(*\omega) \,.$$

Therefore, ω and $*\omega$ have the same energy respect to any observer U and, consequently, both forms have the same energy tensor.

Indeed, it can be checked that:

$$(-1)^{k-1} T(D_1, D_2) = \frac{1}{2}\left(\langle i_{D_1}\omega, i_{D_2}\omega \rangle + (-1)^{n+1} \langle i_{D_1} * \omega, i_{D_2} * \omega \rangle \right) \,.$$

Let us write $T(g,\omega)$ to indicate that the energy tensor depends on the metric g and on the k-form ω. The following three properties will suffice to characterize these energy tensors:

Proposition 5.1.1. *The energy tensor T of a k-form ω satisfies:*

i) For any $\lambda > 0$,

$$T(\lambda^2 g, \lambda^{k-1}\omega) = T(g,\omega) \,.$$

ii) $\operatorname{div} T = i_\omega \mathrm{d}\omega - i_{\partial\omega}\omega$.

iii) At any point $x \in X$,

$$T_x = 0 \quad \Leftrightarrow \quad \omega_x = 0 \,.$$

Proof: The first property easily follows from the definition of energy tensor, and the third one is a consequence that $e(U)(x) = 0$ if and only if $\omega_x = 0$, at any point $x \in X$ and for any observer U.

To compute the divergence of the energy tensor, firstly observe the following local formulae for the components of $i_{\partial\omega}\omega$ and $i_\omega \mathrm{d}\omega$:

$$\begin{aligned}
(i_{\partial\omega}\omega)_b &= \frac{(-1)^k}{(k-1)!} \nabla_a \omega^{a i_2 \dots i_k} \omega_{b i_2 \dots i_k} \,, \\
(i_\omega \mathrm{d}\omega)_b &= \frac{(-1)^k}{k!} \omega^{i_1 \dots i_k} \nabla_b \omega_{i_1 \dots i_k} + \frac{1}{(k-1)!} \omega^{i_1 \dots i_k} \nabla_{i_1} \omega_{i_2 \dots i_k b} \,.
\end{aligned}$$

Then,

$$\begin{aligned}
(\operatorname{div} T)_b &= \nabla_a T^a{}_b = \frac{(-1)^{k-1}}{(k-1)!}\nabla_a\left(\omega^{ai_2\dots i_k}\omega_{bi_2\dots i_k} - \frac{1}{2k}\,\omega^{i_1\dots i_k}\omega_{i_1\dots i_k}\,\delta^a_b\right) \\
&= \frac{(-1)^{k-1}}{(k-1)!}\left((\nabla_a\omega^{ai_2\dots i_p})\omega_{bi_2\dots i_k} + \omega^{ai_2\dots i_k}(\nabla_a\omega_{bi_2\dots i_k}) - \frac{1}{k}\,\omega^{i_1\dots i_k}\nabla_b\omega_{i_1\dots i_k}\right) \\
&= -(i_{\partial\omega}\omega)_b + \frac{1}{(k-1)!}\,\omega^{i_1\dots i_k}\nabla_{i_1}\omega_{i_2\dots i_k b} + \frac{(-1)^k}{k!}\,\omega^{i_1\dots i_k}\nabla_b\omega_{i_1\dots i_k} \\
&= -(i_{\partial\omega}\omega)_b + (i_\omega \mathrm{d}\omega)_b \quad .
\end{aligned}$$

□

The following computation will be used in the proof of Theorem 5.2.3.

Corollary 5.1.2. *For any k-form ω, the 2-covariant tensor $\langle i_-\omega, i_-\omega\rangle$ satisfies:*

$$\operatorname{div}\langle i_-\omega, i_-\omega\rangle = (-1)^{k-1}(i_\omega \mathrm{d}\omega - i_{\partial\omega}\omega) + \frac{1}{2}\,\mathrm{d}\langle\omega,\omega\rangle \ .$$

Proof: By definition of T, we have:

$$\langle i_-\omega, i_-\omega\rangle = (-1)^{k-1}T + \frac{1}{2}\langle\omega,\omega\rangle g \ .$$

Hence:

$$\begin{aligned}
\operatorname{div}\langle i_-\omega, i_-\omega\rangle &= (-1)^{k-1}\operatorname{div} T + \frac{1}{2}\operatorname{div}(\langle\omega,\omega\rangle g) \\
&= (-1)^{k-1}(i_\omega \mathrm{d}\omega - i_{\partial\omega}\omega) + \frac{1}{2}\,\mathrm{d}\langle\omega,\omega\rangle \ .
\end{aligned}$$

□

Remark. Analogously to the case of a 2-form, the energy tensor T of a k-form also appears as the Euler-Lagrange tensor of a variational principle. Namely, if we fix a k-form ω and consider the variational problem of order 0 defined by the lagrangian density $\langle\omega,\omega\rangle\,\omega_X$ on the bundle of Lorentzian metrics, then its Euler-Lagrange equations are precisely $T = 0$.

5.2 Characterization of the energy tensors

Let us analyse separately the consequences of each of the three hypothesis that characterize the energy tensors.

Lemma 5.2.1. *Let $T\colon Metrics \times k\text{-}Forms \longrightarrow 2\text{-}Tensors$ be a natural tensor, with $k \neq 1,3$, that is homogeneous of relative degree $(k-1;0)$; i.e., for all $\lambda \in \mathbb{R}^+$:*

$$T(\lambda^2 g, \lambda^{k-1}\omega) = T(g,\omega) \ .$$

Then T is an $\mathbb{R}$-linear combination of the following four tensors:

$$Ric \quad , \qquad rg \quad , \qquad \langle i_{-}\omega, i_{-}\omega\rangle \quad , \qquad \langle \omega,\omega\rangle g \ .$$

Proof: By Theorem 3.4.2, the space of tensors under consideration is isomorphic to the vector space of O_{g_x}-equivariant linear maps:

$$S^{d_2}N_2 \otimes \ldots \otimes S^{d_r}N_r \otimes S^{\bar{d}_0}\Lambda_0 \otimes \ldots \otimes S^{\bar{d}_s}\Lambda_s \longrightarrow \otimes^2 T^*_x X$$

where $\Lambda_m := \Lambda^k T^*_x X \otimes S^m T^*_x X$, $m = 0,\ldots,s$, and the coefficients $d_i, \bar{d}_j \in \mathbb{N}$ satisfy the equation:

$$2d_2 + \ldots + rd_r + \bar{d}_0 + 2\bar{d}_1 + \ldots + (s+1)\bar{d}_s = 2 \ .$$

If some $\bar{d}_i$ is non zero, then there are only two possibilities:

• $\bar{d}_0 = 2$, $\bar{d}_1 = \ldots = \bar{d}_s = d_j = 0$. In this case, we are reduced to compute O_{g_x}-equivariant linear maps:

$$S^2(\Lambda^k T^*_x X) \otimes T_x X \otimes T_x X \longrightarrow \mathbb{R} \ .$$

Those linear maps are linear combinations of iterated contractions; due to symmetries, any such an iterated contraction is a linear combination of:

$$T^{a_1\ldots a_k a_1 \ldots a_k bb} \qquad , \qquad T^{ba_2\ldots a_k ca_2\ldots a_k bc} \ .$$

These contractions, in turn, correspond, respectively, with the tensors:

$$\langle \omega,\omega\rangle g \quad , \quad \langle i_{-}\omega, i_{-}\omega\rangle \ .$$

- $\bar{d}_1 = 1$, $\bar{d}_0 = \bar{d}_2 \ldots = \bar{d}_s = d_j = 0$: We have to compute O_{g_x}-equivariant linear maps:

$$\Lambda^k T_x^* X \otimes T_x^* X \otimes T_x X \otimes T_x X \longrightarrow \mathbb{R} \; .$$

But there are no such maps for k even or $k \geq 5$ (the contraction of two skew-symmetric indices is zero).

Finally, if the $\bar{d}_i$ are all zero, then the tensor T does not depend on ω and therefore is a linear combination of Ric and rg (Lemma 4.2.1). □

Remark. In the previous Proposition, if $k = 3$ there also exists the O_{g_x}-invariant linear map:

$$\Lambda^3 T_x^* X \otimes T_x^* X \otimes T_x X \otimes T_x X \to \mathbb{R}$$

defined by T_{abcabc}, that corresponds to the natural tensor $c_{01}(\nabla\omega)$, where c_{01} denotes the contraction of the first two indices.

If $k = 1$, there also exist the linear maps:

$$T_x^* X \otimes T_x^* X \otimes T_x X \otimes T_x X \to \mathbb{R}$$

defined by the iterated contractions:

$$T^{aabb} \quad , \quad T^{abab} \quad , \quad T^{abba} \; ,$$

These maps correspond, respectively, with the natural tensors:

$$(\operatorname{div}\omega) g \quad , \quad \nabla\omega \quad , \quad (\nabla\omega)^t$$

where $(\nabla\omega)^t(D_1, D_2) := (\nabla\omega)(D_2, D_1)$.

Lemma 5.2.2. *Let $T\colon Metrics \times k\text{-}Forms \longrightarrow 2\text{-}Tensors$ be a natural tensor satisfying:*

1. *It is homogeneous of relative degree $(k-1; 0)$.*
2. *At any point $x \in X$, if $\omega_x = 0$ then $T_x = 0$.*

Then, T is an $\mathbb{R}$-linear combination of the tensors:

$$\langle i_- \omega, i_- \omega \rangle \quad , \quad \langle \omega, \omega \rangle g \; .$$

Proof: The second condition rules out the tensors Ric and rg in the previous Lemma, as well as the other exceptional tensors in the cases $k = 1,3$. $\square$

Theorem 5.2.3. *If a natural tensor $T\colon Metrics \times k\text{-}Forms \longrightarrow 2\text{-}Tensors$ satisfies:*

1. *It is independent of the unit of scale: $T(\lambda^2 g, \lambda^{k-1}\omega) = T(g,\omega)$ for all $\lambda > 0$,*
2. *At any point, $\quad \omega_x = 0 \;\Rightarrow\; T(g,\omega)_x = 0$,*
3. $\operatorname{div}_g T(g,\omega) = 0$ *whenever ω is closed and co-closed,*

then $T(g,\omega)$ is a constant multiple of the energy tensor:

$$E(g,\omega) := (-1)^{k-1}\left(\langle i_{-}\omega, i_{-}\omega\rangle - \frac{1}{2}\langle\omega,\omega\rangle\, g\right).$$

Proof: By the previous Lemma, there exist universal constants $\mu_1, \mu_2 \in \mathbb{R}$ such that:

$$T(g,\omega) \;=\; \mu_1 \langle i_{-}\omega, i_{-}\omega\rangle + \mu_2 \langle\omega,\omega\rangle\, g\ .$$

Then, writing $T = T(g,\omega)$,

$$\begin{aligned}\operatorname{div} T \;=&\;\; \mu_1 \operatorname{div}(\langle i_{-}\omega, i_{-}\omega\rangle) + \mu_2 \operatorname{div}(\langle\omega,\omega\rangle\, g)\\ \overset{5.1.2}{=}&\;\; \mu_1\left((-1)^{k-1}(i_\omega \mathrm{d}\omega - i_{\partial\omega}\omega) + \frac{1}{2}\mathrm{d}\langle\omega,\omega\rangle\right) + \mu_2 \mathrm{d}\langle\omega,\omega\rangle \quad .\end{aligned}$$

By hypothesis (3), if ω is closed and co-closed, then $\operatorname{div} T = 0$. Comparing with the previous equation, we have

$$0 \;=\; \operatorname{div} T \;=\; \mu_1\left(0 + \frac{1}{2}\,\mathrm{d}\langle\omega,\omega\rangle\right) + \mu_2\, \mathrm{d}\langle\omega,\omega\rangle\ ,$$

hence $\mu_2 = -\mu_1/2$ and we conclude:

$$T \;=\; \mu_1 \langle i_{-}\omega, i_{-}\omega\rangle - \frac{\mu_1}{2}\langle\omega,\omega\rangle\, g \;=\; \mu_1 (-1)^{k-1} E\ .$$

$\square$

We may reformulate the above theorem so as to eliminate the constant factor:

Theorem 5.2.4. *If a natural tensor $T\colon Metrics \times k\text{-}Forms \longrightarrow 2\text{-}Tensors$ satisfies:*

1. *It is independent of the unit of scale: $T(\lambda^2 g, \lambda^{k-1}\omega) = T(g,\omega)$ for all $\lambda > 0$,*

2. *At any point,* $\omega_x = 0 \Rightarrow T(g,\omega)_x = 0$,

3. $\mathrm{div}_g T(g,\omega) = -i_{\partial\omega}\omega$ *whenever* ω *is closed,*

then $T(g,\omega)$ *coincides with the energy tensor* $E(g,\omega)$.

Proof: The tensor T satisfies the hypotheses of the previous theorem, so it is a constant multiple of the energy tensor E. As both tensors have the same divergence whenever ω is closed, that constant has to be one.

□

Finally, let us also remark another possible variation of this result. Let $ClosedForms_k$ be the sheaf of closed k-forms on X.

Theorem 5.2.5. *Let* $T : Metrics \times ClosedForms_k \longrightarrow 2\text{-}Tensors$ *be a natural operator. If it satisfies:*

1. *It is independent of the unit of scale:* $T(\lambda^2 g, \lambda^{k-1}\omega) = T(g,\omega)$ *for all* $\lambda > 0$.

2. *At any point,* $\omega_x = 0 \Rightarrow T(g,\omega)_x = 0$.

3. $\mathrm{div}_g T(g,\omega) = -i_{\partial\omega}\omega$.

then $T(g,\omega)$ *coincides with the energy tensor* $E(g,\omega)$.

Proof: The exterior differential defines a morphism of sheaves:

$$(k-1)\text{-}Forms \xrightarrow{\mathrm{d}} ClosedForms_k$$

that is surjective because any closed form is locally exact.

Hence, any natural operator T as in the statement induces a natural tensor $T' = T \circ \mathrm{d}$:

$$Metrics \times (k-1)\text{-}Forms \xrightarrow{\mathrm{Id}\times\mathrm{d}} Metrics \times ClosedForms_k \xrightarrow{T} 2\text{-}Tensors$$

satisfying the homogeneity condition:

$$T'(\lambda^2 g, \lambda^{k-1}\omega_{k-1}) = T'(g,\omega_{k-1}) , \quad \forall\, \lambda \in \mathbb{R}^+ , \tag{5.2.0.3}$$

and other two hypothesis, corresponding to items *2* and *3* of the statement:

$$\mathrm{d}_x\omega = 0 \Rightarrow T'(g,\omega)_x = 0 , \qquad \text{and} \qquad \mathrm{div}_g T'(g,\omega) = -i_{\partial\mathrm{d}\omega}\mathrm{d}\omega .$$

To finish the proof, it is enough to prove that $T' = T \circ \mathrm{d}$ is the tensor:

$$T'(g,\omega) = (-1)^{k-1}\left(\langle i_{-}\mathrm{d}\omega\,,\, i_{-}\mathrm{d}\omega\rangle - \frac{1}{2}\langle \mathrm{d}\omega\,,\,\mathrm{d}\omega\rangle\, g\right)\ ,$$

so that the original operator $T(g,\omega)$ coincides with the energy tensor.

To begin with, it is not difficult to check that the natural tensors $T' = T \circ \mathrm{d}$ satisfying (5.2.0.3) are in bijection with O_{g_x}-equivariant linear maps:

$$S^{d_2}N_2 \otimes \ldots \otimes S^{d_r}N_r \otimes S^{\bar{d}_0}\Lambda_0 \otimes \ldots \otimes S^{\bar{d}_s}\Lambda_s \longrightarrow \otimes^2 T^*_x X$$

where $\Lambda_m := \Lambda^k T^*_x X \otimes S^m T^*_x X$, for $m = 1,\ldots,s$ (observe the spaces of *k-forms* at the point, not $(k-1)$-forms), and the non-negative integers d_i, $\bar{d}_j$ satisfy the equation:

$$2d_2 + \ldots + r d_r + \bar{d}_0 + \ldots + (s+1)\bar{d}_s = 2\ . \tag{5.2.0.4}$$

Now, the computations proceed in analogy with those in Lemma 5.2.1 and Theorem 5.2.4. If some $\bar{d}_i$ is non zero, then there are only two possibilities:

• $\bar{d}_0 = 2$, $\bar{d}_1 = \ldots = \bar{d}_s = d_j = 0$. In this case, a similar analysis to that in page 74 shows that the only natural tensors are the linear combinations of:

$$\langle \mathrm{d}\omega\,,\,\mathrm{d}\omega\rangle\, g \quad , \quad \langle i_{-}\mathrm{d}\omega\,,\, i_{-}\mathrm{d}\omega\rangle$$

• $\bar{d}_1 = 1$, $\bar{d}_0 = \bar{d}_2 \ldots = \bar{d}_s = d_j = 0$: For this solution, the only possible tensors are $c_{01}(\nabla \mathrm{d}\omega)$, if $k = 3$, and $(\mathrm{div}(\mathrm{d}\omega))g$, $\nabla \mathrm{d}\omega$, $(\nabla \mathrm{d}\omega)^t$, if $k = 3$.

The hypothesis number *2* rules out these last tensors, as well as those not explicitly depending on ω.

That is, there exist $\lambda, \mu \in \mathbb{R}$ such that the tensor $T' = T \circ \mathrm{d}$ is of the form:

$$T' = \lambda\langle \mathrm{d}\omega\,,\,\mathrm{d}\omega\rangle\, g + \mu\langle i_{-}\mathrm{d}\omega\,,\, i_{-}\mathrm{d}\omega\rangle\ .$$

Applying hypothesis number *3*, we can fix the scalars:

$$\begin{aligned}
-i_{\partial \mathrm{d}\omega}\mathrm{d}\omega = \mathrm{div}_g\left(T'(g,\omega)\right) &= \lambda\,\mathrm{d}(\langle \mathrm{d}\omega\,,\,\mathrm{d}\omega\rangle) + \mu\,\mathrm{div}_g(\langle i_{-}\mathrm{d}\omega\,,\, i_{-}\mathrm{d}\omega\rangle) \\
&\overset{5.1.2}{=} \lambda\,\mathrm{d}(\langle \mathrm{d}\omega\,,\,\mathrm{d}\omega\rangle) + \mu(-1)^k\, i_{\partial \mathrm{d}\omega}\mathrm{d}\omega + \frac{\mu}{2}\mathrm{d}\langle \mathrm{d}\omega, \mathrm{d}\omega\rangle\ , \\
&= \left(\lambda + \frac{\mu}{2}\right)\mathrm{d}(\langle \mathrm{d}\omega\,,\,\mathrm{d}\omega\rangle) + \mu(-1)^k\, i_{\partial \mathrm{d}\omega}\mathrm{d}\omega\ ,
\end{aligned}$$

that implies $\lambda = -\frac{\mu}{2}$ and $\mu = (-1)^{k-1}$.

$\square$

5.3 Electromagnetism of branes

There exists a theory of electromagnetism for charged p-branes ([15]), where the electromagnetic field is represented by a differential $(p+2)$-form F. In the rest of the Chapter, we shall extend this theory up to the point of including a force law for fluids of charged p-branes and an electromagnetic energy tensor, necessary to state the Einstein equation. This tensor is precisely the energy tensor of the form F introduced in page 70.

But firstly, let us analyse in this Section the interaction of a charged p-brane with an arbitrary electromagnetic field. Our analysis is developed at the classical (non quantum) level and, in contrast to [15], it is based on the elementary concepts of impulse and acceleration of a p-brane.

From now on, let us fix an integer p, such that $0 \leq p \leq n$, and let us write $q := n - p$.

Definition. The trajectory of a p-brane is, by definition, an oriented smooth submanifold $S \subset X$ of dimension $p+1$, whose metric $g_{|S}$ has signature $(+,-,\overset{p}{\ldots},-)$.

Associated to any p-brane, we also assume two constants, called tension $\mathfrak{t} > 0$ (or mass, in the case $p = 0$ of punctual particles), and electric charge $\mathfrak{q} \in \mathbb{R}$.

5.3.1 Impulse of a brane

In absence of external forces, the trajectory of a punctual particle is a geodesic of spacetime. To extend this fundamental principle to the movement of a p-brane, let us recall two different characterizations of geodesics:

1. The trajectory of a particle is a geodesic if the impulse vector mU is parallel along the trajectory (where m is the mass of the particle and U is the future pointing, unitary tangent vector to the trajectory).

2. The trajectory of a particle is a geodesic if it minimizes the action $m\int \mathrm{d}\tau$, where τ is the proper time of the trajectory.

In the literature, in order to determine the movement of a p-brane in absence of external forces, it is common to follow the second approach, using variational principles. To be precise, the generalized action is the Nambu-Goto action, $\mathfrak{t}\int_S \omega_S$, where ω_S is the $(p+1)$-volume of the trajectory S of the brane.

Instead of that, let us generalize the concept of impulse to a p-brane, arriving to the same equations of motion.

Let $S \subset X$ be the trajectory of a p-brane and let ω_S be the $(p+1)$-volume form of S. Rising the first index of ω_S and multiplying it by the tension $\mathfrak{t}$, we obtain a p-form with values on tangent vectors, that is called the *impulse form* of S:

Definition. The impulse form of a p-brane S is the p-form on S with values on TS:

$$\Pi_S \,:\, TS \wedge \overset{p}{\ldots} \wedge TS \longrightarrow TS \subset (TX)_{|S}$$

defined by the following property:

$$g(D_0, \Pi_S(D_1, \ldots, D_p)) = \mathfrak{t}\,\omega_S(D_0, \ldots, D_p)$$

for any $D_0, \ldots, D_p$ tangent vectors to S.

If $D_0, \ldots, D_p$ is an orthonormal frame of vector fields on S, where the matrix of $g_{|S}$ is diagonal (+1, -1, ..., -1), then:

$$\Pi_S = \mathfrak{t} \sum_{j=0}^{p} (i_{D_j}\omega_S) \otimes \delta_j D_j$$

where $\delta_0 = 1$ and $\delta_j = -1$ for $j \neq 0$.

Examples. In the case of a particle ($p = 0$), the trajectory S is a curve and the impulse form Π_S is a vector-valued 0-form; that is, it is simply a tangent vector $\Pi_S = mU$, where m is the mass of the particle and U is the unitary tangent vector to the curve.

If $(X = \mathbb{R}^{1+n}, g = \mathrm{d}t^2 - \sum_i \mathrm{d}x_i^2)$ is the Minkowski spacetime, then the impulse form of a p-brane S can be written as:

$$\Pi_S = \omega_0 \otimes \partial_t + \omega_1 \otimes \partial_{x_1} + \ldots + \omega_n \otimes \partial_{x_n}$$

for some ordinary differential p-forms ω_i on S.

If $S_{t_0} := S \cap \{t = t_0\}$ is the particle at the instant t_0, then the vector:

$$\int_{S_{t_0}} \Pi_S := \left(\int_{S_{t_0}} \omega_0 \right) \partial_t + \ldots + \left(\int_{S_{t_0}} \omega_n \right) \partial_{x_n}$$

can be understood as the total energy-impulse vector of the p-brane at t_0.

The differential p-form ω_0 is called *energy form* of the brane respect to the chosen inertial frame $(t, x_1, \ldots, x_n)$, and the integral $\int_{S_{t_0}} \omega_0$ is understood as the total energy of the brane at t_0.

5.3.2 Acceleration of a brane

Let $S \subset X$ be the trajectory of a p-brane, and let us write ∇ for the Levi-Civita connection of (X, g). For any pair of tangent vector fields D, D' on S, the covariant derivative $D^{\nabla}D'$ decomposes as a tangent vector to S plus a vector orthogonal to S:

$$D^{\nabla}D' = \operatorname{tang}(D^{\nabla}D') + \operatorname{nor}(D^{\nabla}D') \ .$$

The first addend $D^{\overline{\nabla}}D' := \operatorname{tang}(D^{\nabla}D')$ is precisely the covariant derivative with respect to the Levi-Civita connection $\overline{\nabla}$ of the submanifold $(S, g_{|S})$. The second addend is, by definition, the *second fundamental form* of S, which is a symmetric tensor with values on the normal bundle of S:

$$\Phi_S : TS \times TS \longrightarrow (TS)^{\perp} \quad , \quad \Phi_S(D, D') := \operatorname{nor}(D^{\nabla}D') \ .$$

The trace of the second fundamental form, $\operatorname{tr}\Phi_S$, is a field of normal vectors to S.

Proposition 5.3.1. *The impulse form Π_S of a p-brane S satisfies:*

$$\mathrm{d}_{\nabla}\Pi_S = \omega_S \otimes \mathfrak{t} \cdot \operatorname{tr}(\Phi_S) \ .$$

Proof: Let $(D_0, \ldots, D_n)$ be an orthonormal basis of vector fields on X, such that $(D_0, \ldots, D_p)$ is an orthonormal basis of vector fields on S.

Let ω_{ij} be the connection 1-forms, so that $\mathrm{d}_{\nabla} D_j = \sum_{i=0}^{n} \omega_{ij} \otimes D_i$ and hence:

$$D_j^{\nabla} D_j = \sum_{i=0}^{n} \omega_{ij}(D_j) D_i \quad \text{ and } \quad D_j^{\bar{\nabla}} D_j = \sum_{i=0}^{p} \omega_{ij}(D_j) D_i \ , \quad j \le p.$$

Let Π_S and T_S be the impulse form and the energy-momentum tensor of S, respectively. Locally:

$$\Pi_S = \mathfrak{t} \sum_{i=0}^{p} (i_{D_i}\omega_S) \otimes \delta_i D_i \quad , \quad T_S = \mathfrak{t} \sum_{i=0}^{p} D_i \otimes D_i$$

where we write $\delta_0 = 1$, $\delta_i = -1$ for $i \neq 0$.

When considered as tensors with values on $TX_{|S}$, both T_S and Π_S correspond via the isomorphism of Proposition 4.1.1. Therefore, if $\tilde{\bar{\nabla}}$ denotes the connection induced on $TS \otimes TX_{|S}$ by the pair of connections $\bar{\nabla}$ and ∇, we only have to check that $\mathrm{div}_{\tilde{\bar{\nabla}}} T_S = \mathrm{tr}\,\Phi_S$:

$$\mathrm{d}_{\tilde{\bar{\nabla}}} T_S = \sum_{j=0}^{p} \mathrm{d}_{\bar{\nabla}} D_j \otimes D_j + \sum_{j=0}^{p} D_j \otimes \mathrm{d}_{\nabla} D_j = \sum_{i,j=0}^{p} \omega_{ij} \otimes D_i \otimes D_j + \sum_{j=0}^{p} D_j \otimes \sum_{i=0}^{n} \omega_{ij} \otimes D_i \ .$$

$$\mathrm{div}_{\tilde{\bar{\nabla}}} T_S = c_1^1(\mathrm{d}_{\tilde{\bar{\nabla}}} T_S) = -\sum_{i=0}^{p}\left(\sum_{j=0}^{p} \omega_{ji}(D_i) D_j\right) + \sum_{j=0}^{p}\left(\sum_{i=0}^{n} \omega_{ij}(D_j) D_i\right)$$

$$= -\sum_{i=0}^{p} D_i^{\bar{\nabla}} D_i + \sum_{j=0}^{p} D_j^{\nabla} D_j = \sum_{j=0}^{p}\left(D_j^{\nabla} D_j - D_j^{\bar{\nabla}} D_j\right) = \sum_{j=0}^{p} \Phi(D_j, D_j) = \mathrm{tr}\,\Phi_S \ .$$

□

Example. If $p = 0$, let S be the trajectory of a particle with impulse $\Pi_S = mU$, where m is the mass of the particle and U is the future-pointing unitary tangent vector of the curve S. Since:

$$\Phi_S(U,U) = \mathrm{nor}(U^{\nabla} U) = U^{\nabla} U$$

we observe $\mathrm{tr}\,\Phi_S = U^{\nabla} U$ is the acceleration of the particle.

Definition. By analogy with the particle case just explained, if S is the trajectory of a p-brane, then the normal vector $\mathrm{tr}\,\Phi_S$ is interpreted as the acceleration of the brane.

If there are no external forces, the trajectory S of a brane should have null acceleration. For a particle, this amounts to saying that it is a geodesic: $U^{\nabla} U = 0$. For a p-brane, this amounts to the equation:

$$\textbf{Inertial Motion:} \quad \boxed{\mathrm{tr}\,\Phi_S = 0} \ .$$

By Proposition 5.3.1, this equation is equivalent to $\mathrm{d}_{\nabla} \Pi_S = 0$, which is an infinitesimal conservation law for the impulse.

Moreover, this equation $\mathrm{tr}\,\Phi_S = 0$ is precisely the Euler-Lagrange equation for the variational problem defined by the Nambu-Goto action.

5.3.3 Electromagnetic field

A distribution of charged p-branes produces an *electromagnetic field* over the spacetime X. To specify this field we have to say, at any point x, the acceleration measured by a brane S with unitary charge, which is a vector field orthogonal to the brane.

For simplicity, we assume this assignment is linear, thus arriving to the following definition:

Definition. An electromagnetic field over the spacetime X is a skew-symmetric tensor:

$$\widehat{F}: TX \wedge \overset{p+1}{\cdots\cdots} \wedge TX \longrightarrow TX$$

satisfying the following property:

$$\widehat{F}(D_0,\ldots,D_p) \in < D_0,\ldots,D_p >^{\perp}$$

for any collection $D_0,\ldots,D_p$ of vector fields on X.

The value $\widehat{F}(D_0,\ldots,D_p)_x$ may be understood as the force at the point x that suffers a brane with $(p+1)$-volume vector $D_0 \wedge \ldots \wedge D_p$ and unitary charge (see the force law below).

The definition of $\widehat{F}$ amounts to saying that the tensor:

$$F(D_0,\ldots,D_{p+1}) := g(\widehat{F}(D_0,\ldots,D_p), D_{p+1})$$

is a $(p+2)$-differential form on X, and we will say that F is the $(p+2)$-form of the electromagnetic field.

Force Law for a p-brane

Let S be the trajectory of a p-brane with tension $\mathfrak{t}$ and electric charge $\mathfrak{q}$.

Definition. The charge-current vector of this brane is the only $(p+1)$-vector J_S on S satisfying

$$\omega_S(J_S) = \mathfrak{q} \ .$$

If $(D_0,\ldots,D_p)$ is an oriented orthonormal frame of vector fields on S, then:

$$J_S = \mathfrak{q} D_0 \wedge \ldots \wedge D_p \ .$$

Let $\widehat{F}$ be an electromagnetic force and assume that the p-brane S does not substantially modify the electromagnetic field. Nevertheless, the p-brane S does suffer an acceleration due to the force $\widehat{F}$, that we postulate to be governed by the following equation:

$$\textbf{Lorentz Force Law}: \quad \boxed{\mathrm{d}_\nabla \Pi_S = \omega_S \otimes \widehat{F}(J_S)} \ .$$

Using Proposition 5.3.1, this equation is equivalent to $\mathfrak{t}\cdot\operatorname{tr}\Phi_S = \widehat{F}(J_S)$, which, substituting the value of J_S, is in turn equivalent to:

$$\mathfrak{t}\cdot\operatorname{tr}\Phi_S = \mathfrak{q}\cdot\widehat{F}(D_0,\ldots,D_p)\ .$$

Observe the typical form of this equation: $mass \times acceleration = force$; the definition of $\widehat{F}$ has been dictated by the need of giving sense to this expression.

Examples. In the case $p = 0$, the charge-current vector of a particle is simply a vector $J_S = \mathfrak{q}U$, where U is the future-pointing, unitary tangent vector of the trajectory S of the particle.

Since the impulse of the particle is mU, the force law reads:

$$\mathrm{d}_\nabla(mU) = \mathrm{d}\tau\otimes\mathfrak{q}\widehat{F}(U)$$

where τ stands for the proper time of the curve.

As $\mathrm{d}_\nabla U = \mathrm{d}\tau\otimes\partial_\tau^\nabla U$, this force law is equivalent to the equation:

$$m\cdot\partial_\tau^\nabla U = \mathfrak{q}\cdot\widehat{F}(U)$$

which is precisely the classical Lorentz Force Law for a particle of mass m and charge $\mathfrak{q}$.

Let $(X = \mathbb{R}^{n+1}, g = \mathrm{d}t^2 - \sum_1^n \mathrm{d}x_i^2)$ be the Minkowski spacetime. The trajectory of a brane S can be written as $x_i = f_i(t,u_1,\ldots,u_p)$, where $(t,u_1,\ldots,u_p)$ are local coordinates on S.

On these coordinates $(t,u_1,\ldots,u_p)$, the force law $\mathfrak{t}\cdot\operatorname{tr}\Phi = \widehat{F}(J_S)$ produces a system of second order partial differential equations, that is quasi-linear and hyperbolic.

For these kind of systems, the Cauchy problem has a unique local solution, so the force law uniquely determines the trajectory of the p-brane, for adequate initial conditions.

5.3.4 Maxwell equations

Definition. A distribution of charged p-branes on the spacetime X will be represented by means of a differential q-form C, called the charge-density form.

The intuitive interpretation of this q-form is the following: given $q := n - p$ linearly independent vectors $D_1,\ldots,D_q \in T_xX$, that we understand as an oriented infinitesimal parallelogram at the point x, we have:

$$C(D_1,\ldots,D_q) = \left\{ \begin{array}{c} \text{Sum, affected with a sign, of the charges of the p–branes} \\ \text{transversally crossing the parallelogram } D_1,\ldots,D_q \end{array} \right\}.$$

We say that a p-brane with a trajectory S transversally crosses the parallelogram $D_1,\ldots,D_q$ whenever $T_xX = T_xS \oplus \langle D_1,\ldots,D_q\rangle$. If the orientation of T_xX coincides with the product of the orientations on T_xS and $\langle D_1,\ldots,D_q\rangle$, then the charge of the p-brane counts with positive sign; otherwise, we affect the charge with a negative sign.

Definition. The charge-current $(p+1)$-vector of a distribution of charged p-branes is the only $(p+1)$-vector J satisfying:

$$i_J\omega_X = C\ .$$

Equivalently, if J^* is the $(p+1)$-form metrically equivalent to J and $*$ stands for the Hodge operator,

$$J^* = (-1)^{pn} * C\ .$$

Example. When $p = 0$, the charge-density form C is a differential n-form, and the charge-current vector J is simply a vector field on X.

In this case, the electromagnetic field F is a 2-form, related to the distribution of charges by the Maxwell equations:

$$\mathrm{d}F = 0 \qquad , \qquad \partial F = J^*\ .$$

Let us consider a distribution of charged p-branes, represented by a charge-density q-form C or, equivalently, by a charge-current $(p+1)$-vector J. Such a distribution of charges "produces" an electromagnetic field, represented by a $(p+2)$-form F. By analogy with the particle case, we postulate that both fields are related by the following:

Maxwell Equations: $\boxed{\mathrm{d}F = 0 \quad , \quad \partial F = J^*}$

or equivalently

$$\mathrm{d}F = 0 \quad , \quad \mathrm{d}(*F) = (-1)^p C\ .$$

The second Maxwell equation implies an infinitesimal charge conservation law $\mathrm{d}C = 0$ (equivalently, $\partial J^* = 0$). The operators d and ∂ are, essentially, the only first-order natural linear differential operators between differential forms. Therefore, in a certain sense, Maxwell equations are the only possible first-order equations that may arise.

Remark. In a similar vein to what is done for charged particles ($p = 0$), the Lorentz force law and the Maxwell equations may be derived from variational principles: let us write $F = \mathrm{d}A$ where A is a $(p+1)$-form on spacetime, called the *electromagnetic potential*. For each trajectory S of a p-brane with tension $\mathfrak{t}$ and electric charge $\mathfrak{q}$, consider the action:

$$\mathscr{A}(S) := -\mathfrak{t}\int_S \omega_S + (-1)^p \mathfrak{q}\int_S A \ .$$

Extremals of this action are precisely the trajectories that satisfy the Lorentz force law.

On the other hand, let us fix a closed q-form C on the spacetime X. For any $(p+1)$-form A, consider the action:

$$\mathscr{A}(A) := \int_X \frac{1}{2} F \wedge *F - \int_X A \wedge C \ ,$$

where $F := \mathrm{d}A$. The Euler-Lagrange equations for this action amount to the Maxwell equation $\partial F = J^*$, where $i_J \omega_X = C$.

5.4 Fluids of charged branes

Now we shall extend the notion of impulse of a p-brane and the force law to the case of a fluid of charged p-branes. We shall also define the electromagnetic energy tensor associated to the electromagnetic field strength F, which is necessary to formulate the Einstein equation.

5.4.1 Impulse form and force law

Definition. The energy-momentum distribution of a fluid of charged p-branes is represented by a differential n-form Π_n with values on TX, called the impulse form of the fluid.

The 2-covariant tensor T_2 corresponding to Π_n via the isomorphism of Proposition 4.1.1 is called the energy-momentum tensor of the fluid.

The interpretation of the impulse n-form Π_n is the following: assume the ambient manifold X is the Minkowski spacetime and let H be an oriented hypersurface.

If S is the trajectory of a p-brane transversally crossing the hypersurface H, and Π_S is the vector-valued impulse p-form of the p-brane, then the vector $\int_{S\cap H} \Pi_S$ is said to be the *total impulse* of the particle in the hypersurface H (see Example in page 80).

Now, the vector $\int_H \Pi_n$ is understood as the sum of total impulses of all the charged p-branes transversally crossing the hypersurface H.

Let us consider a fluid of charged p-branes, with impulse form Π_n and charge-current $(p+1)$-vector J.

In absence of external forces, the variation of the impulse should be null: $\mathrm{d}_\nabla \Pi_n = 0$. By analogy with the case of a single brane, we postulate that, in presence of an electromagnetic field, the movement of the fluid satisfies the Force Law:

$$\mathrm{d}_\nabla \Pi_n = \omega_X \otimes \widehat{F}(J) .$$

In virtue of Proposition 4.1.1, this equation is equivalent to $\operatorname{div} T^2 = \widehat{F}(J)$, or to $\operatorname{div} T_2 = i_J F$. Combining it with the Maxwell equation $\partial F = J^*$, we obtain another equivalent formulation:

$$\boxed{\operatorname{div} T_2 = i_{\partial F} F} \; .$$

An example: dusts of charged branes

Let us consider a fluid of charged branes without pressure and where all the branes in the fluid have the same tension t and the same electric charge q. The general idea is that each brane has approximately the same velocity as the surrounding ones; hence, we give the following definition:

A *dust* of p-branes is described by an integrable distribution on X of rank $p+1$, for which each integrable submanifold represents the mean trajectory of an infinitesimal portion of p-branes.

Let $(D_0,\dots,D_p)$ be an orthonormal basis $(+,-,\dots,-)$ of the distribution. Such a basis defines an orientation on each integral submanifold. By analogy with the case of a single p-brane, the charge-current $(p+1)$-vector of the dust is defined as:

$$J := \rho_e D_0 \wedge \ldots \wedge D_p \ ,$$

for some *charge density* function ρ_e.

The contravariant stress-energy tensor of the dust is defined by the formula:

$$T^2 := \rho_m \sum_{j=0}^{p} \delta_j D_j \otimes D_j$$

where $\delta_0 = 1$, $\delta_{j\neq 0} = -1$, and the function $\rho_m := (\mathrm{t}/\mathrm{q})\rho_e$ is called the *mass density* (that is, on each trajectory S of the dust we consider the dual metric $(g_{|S})^*$ multiplied by the function ρ_m). According 4.1.1, the corresponding impulse form is:

$$\Pi_n = \rho_m \sum_{j=0}^{p} (i_{D_j}\omega_X) \otimes \delta_j D_j \ .$$

Proposition 5.4.1. *If the charge conservation law* $\mathrm{d}C = 0$ *holds, then the impulse form of a charged dust satisfies:*

$$\mathrm{d}_\nabla \Pi_n = \rho_m \, \omega_X \otimes \operatorname{tr} \Phi$$

where Φ *is the second fundamental form of the trajectories of the dust.*

Proof: Let us complete the orthonormal basis $(D_0, \dots, D_p)$ of the distribution up to an oriented orthonormal basis $(D_0, \dots, D_n)$ of tangent fields on X, and let $(\theta_0, \dots, \theta_n)$ be the corresponding dual basis of 1-forms.

The charge-density q-form is:

$$C = i_J \omega_X = i_J(\theta_0 \wedge \cdots \wedge \theta_n) = \rho_e \, \theta_{p+1} \wedge \cdots \wedge \theta_n \; .$$

We have:

$$\begin{aligned} \Pi_n &= \rho_m \sum_{j=0}^{p} (i_{D_j} \omega_X) \otimes \delta_j D_j = \rho_m \sum_{j=0}^{p} i_{D_j}(\theta_0 \wedge \cdots \wedge \theta_p) \wedge (\theta_{p+1} \wedge \cdots \wedge \theta_n) \otimes \delta_j D_j \\ &= \left(\sum_{j=0}^{p} i_{D_j}(\theta_0 \wedge \cdots \wedge \theta_p) \otimes \delta_j D_j \right) \wedge (\rho_m \, \theta_{p+1} \wedge \cdots \wedge \theta_n) \; . \end{aligned}$$

and therefore,

$$\begin{aligned} \mathrm{d}_\nabla \Pi_n &= \left(\mathrm{d}_\nabla \sum_{j=0}^{p} i_{D_j}(\theta_0 \wedge \cdots \wedge \theta_p) \otimes \delta_j D_j \right) \wedge (\rho_m \, \theta_{p+1} \wedge \cdots \wedge \theta_n) \\ &\quad + (-1)^p \left(\sum_{j=0}^{p} i_{D_j}(\theta_0 \wedge \cdots \wedge \theta_p) \otimes \delta_j D_j \right) \wedge \mathrm{d}(\rho_m \, \theta_{p+1} \wedge \cdots \wedge \theta_n) \; . \end{aligned}$$

The second addend is null because $\mathrm{d}(\rho_m \, \theta_{p+1} \wedge \cdots \wedge \theta_n) = \mathrm{d}(\frac{\mathrm{t}}{\mathrm{q}} C) = 0$. With respect to the first one, the term which is differentiated has the same expression than the impulse form of each integral submanifold (considered as the trajectory S of a p-brane of tension 1). Applying Proposition 5.3.1, we obtain:

$$\begin{aligned} \mathrm{d}_\nabla \Pi_n &= \left(\mathrm{d}_\nabla \sum_{j=0}^{p} i_{D_j}(\theta_0 \wedge \cdots \wedge \theta_p) \otimes \delta_j D_j \right) \wedge (\rho_m \, \theta_{p+1} \wedge \cdots \wedge \theta_n) \\ &= \big(\theta_0 \wedge \cdots \wedge \theta_p \otimes \operatorname{tr} \Phi \big) \wedge (\rho_m \, \theta_{p+1} \wedge \cdots \wedge \theta_n) = \rho_m \omega_X \otimes \operatorname{tr} \Phi \; . \end{aligned}$$

□

As a consequence, if the electromagnetic field F satisfies the Maxwell equations (so, in particular, the charge conservation law holds), then the force law for a dust $\operatorname{div} T^2 = \widehat{F}(J)$ is equivalent to:

$$\rho_m \operatorname{tr}\Phi = \widehat{F}(J) \ ,$$

which is consistent with the force law $\mathfrak{t}\cdot\operatorname{tr}\Phi = \widehat{F}(J_S)$ for each test particle of the dust.

5.4.2 Electromagnetic energy tensor

Definition. Let F be an electromagnetic field. Its electromagnetic energy tensor is the 2-covariant tensor T_{elm} associated to the $(p+2)$-differential form F according to the definition in page 69.

Let F be the electromagnetic field produced by a fluid of charged p-branes with stress-energy tensor T_{m}. The Lorentz force law $\operatorname{div} T_{\text{m}} = i_{\partial F} F$ and Proposition 5.1.1 produce an infinitesimal conservation law:

$$\operatorname{div}(T_{\text{m}} + T_{\text{elm}}) = i_{\partial F} F + (-i_{\partial F} F) = 0 \ .$$

Indeed, this property is the main motivation for the definition of the electromagnetic energy tensor.

Finally, as in the particle case, we postulate that the electromagnetic energy has a gravitational effect through the Einstein equation.

To sum up, a *fluid of charged p-branes* is described by four tensor fields on spacetime: an energy-momentum tensor T_m and a charge-current $(p+1)$-vector J representing the distributions of mass and charge, and a differential $(p+2)$-form F and its energy tensor T_{elm}, representing the electromagnetic field and its electromagnetic energy.

They are related by the following equations:

Maxwell equations: $\quad \mathrm{d}F = 0 \quad , \quad \partial F = J^* \ .$

Einstein equation: $\quad Ric - \dfrac{r}{2} g = T_m + T_{\text{elm}} \quad .$

Chapter 6

Second-order, divergence-free tensors

In this Chapter we consider the problem of describing natural tensors associated to a metric that are divergence-free. We restrict our attention to second-order tensors (i.e. tensors whose coefficients involve second derivatives of the metric only) but, apart from this, we consider an arbitrary number of indices and symmetries among them.

The main outcome of our investigations is a procedure to describe the vector space of second order p-tensors that are divergence-free as a certain space of tensors invariant under the action of the orthogonal group.

The first step in this procedure is to prove that tensors with zero divergence are "algebraic", in the sense that their coefficients are polynomial functions of the second derivatives of the metric:

Theorem 6.2.4: *Any second-order, natural tensor that is divergence-free is polynomial, of degree* $\leq (n-1)/2$ *in the second derivatives of the metric (* $n = \dim X$*).*

In particular, the vector space of second-order, natural tensors that are divergence-free is finite dimensional.

The proof of this statement relies on some techniques introduced by Lovelock, that we develop so as to show the vanishing of derivatives of sufficiently large order. In the process, we make use of simple facts of graph theory, that allows to avoid lengthy calculations using symmetries of indices.

Secondly, it is easy to see that a polynomial tensor is divergence-free if and only if its homogeneous components are divergence-free (Proposition 6.3.2).

Then, we introduce certain spaces of tensors, Div_x^k, depending on the symmetries and the number p of indices of the tensors under consideration, and we prove:

Theorem 6.3.1 *Let g_x be a pseudo-Riemannian metric at a point $x \in X$ with the fixed signature and let O_{g_x} be the orthogonal group of (T_xX, g_x).*

For any $m \geq 1$ there exists an injective map:

$$\left[\begin{array}{c} \textit{Divergence-free, second-order, natural } p\textit{-tensors} \\ \textit{homogeneous of degree } m \textit{ in the second derivatives} \end{array} \right] \subseteq \left(\mathrm{Div}_x^m\right)^{O_{g_x}} .$$

Altogether, these results reduce the original question to computing the space $\left(\mathrm{Div}_x^m\right)^{O_{g_x}}$ of invariant tensors under the action of the orthogonal Lie group O_{g_x}, where the classical theory of invariants can be applied.

In the last Sections of the Chapter, we apply this technique to compute basis for these spaces of divergence-free tensors in some particular cases.

In the simplest situation, that of tensors with $p = 2$ indices, we recover the celebrated Lovelock's result. Indeed, we prove a stronger statement, as we show that the Lovelock's tensors are also a basis for the vector space of second-order, divergence-free 2-tensors, but not necessarily symmetric:

Theorem 6.4.3: *The Lovelock tensors $L_0, \ldots, L_m$, where $2m \leq n-1$, are a basis for the $\mathbb{R}$-vector space of second-order, natural 2-tensors that are divergence-free.*

In dimension 4, this refined version was already established by Lovelock himself ([31]), but the situation in higher dimensions remained open.

Finally, as regards to tensors with some of their indices symmetric, we recover some of the results of [8] and [7]. We also prove a new statement on the non-existence of skew-symmetric tensors with zero divergence.

Some of the original results of this Chapter have been published in [37].

6.1 Derivative of second-order tensors

Let $M \to X$ be the bundle of pseudo-Riemannian metrics with a fixed signature. For simplicity, from now on we will only consider *contravariant* tensors, in contrast to the previous Chapters:

Definition. A second-order p-tensor (not necessarily natural) is a second-order differential operator $T \colon M \rightsquigarrow \otimes^p TX$, that is, a morphism of bundles:

$$T \colon J^2M \to \otimes^p TX .$$

In local coordinates, the components of a second-order p-tensor are smooth functions

$$T(g)^{j_1 \dots j_p} = T^{j_1 \dots j_p}(x_i, g_{ab}, g_{ab,c}, g_{ab,cd}) .$$

Definition. The divergence of a p-tensor T on a pseudo-Riemannian manifold (X, g) is the $(p-1)$-tensor:

$$\operatorname{div}_g T := c_1^p(\nabla_g T) ,$$

and its local expression, using summation over repeated indices, is:

$$(\operatorname{div}_g T)^{j_1 \dots j_{p-1}} = \nabla_k T^{j_1 \dots j_{p-1} k} .$$

The divergence of a second-order p-tensor $T \colon M \rightsquigarrow \otimes^p TX$ is the third order differential operator $\operatorname{div} T \colon M \rightsquigarrow \otimes^{p-1} TX$ defined as:

$$(\operatorname{div} T)(g) := \operatorname{div}_g(T(g)) .$$

A second-order p-tensor T is divergence-free if $\operatorname{div} T$ is the zero map.

Proposition 6.1.1. *If a second-order p-tensor T is divergence-free, then, on any chart, the functions $T^{j_1 \dots j_p}$ satisfy:*

$$\frac{\partial T^{j_1 \dots j_p}}{\partial g_{ab,cd}} + \frac{\partial T^{j_1 \dots d}}{\partial g_{ab,j_p c}} + \frac{\partial T^{j_1 \dots c}}{\partial g_{ab,d j_p}} = 0 .$$

Proof: In local coordinates:

$$\begin{aligned}(\operatorname{div} T)^{j_1 \dots j_{p-1}} &= \frac{\partial T^{j_1 \dots k}}{\partial x^k} + \Gamma^{j_1}_{sk} T^{s \dots k} + \dots + \Gamma^{k}_{sk} T^{j_1 \dots s} \\ &= \sum_{a \le b} \sum_{c \le d} \sum_{k=0}^{n} \frac{\partial T^{j_1 \dots k}}{\partial g_{ab,cd}} g_{ab,cdk} + F(x_i, g_{ab}, g_{ab,c}, g_{ab,cd})\end{aligned}$$

for some smooth function F on $J^2 M$.

Reordering this sum, we obtain:

$$\sum_{a \le b} \sum_{c \le d \le k} \left(\frac{\partial T^{j_1 \dots k}}{\partial g_{ab,cd}} + \frac{\partial T^{j_1 \dots d}}{\partial g_{ab,ck}} + \frac{\partial T^{j_1 \dots c}}{\partial g_{ab,dk}} \right) g_{ab,cdk} + F(x_i, g_{ab}, g_{ab,c}, g_{ab,cd}) ,$$

and the thesis follows because $g_{ab,cdk}$ are elements of a chart on $J^3 M$.

$\square$

The fibre $\mathbb{A}_{j^1_x g}$ of the projection $J^2_x M \to J^1_x M$ on $j^1_x g$ is an affine space, modelled on the vector space $S_{2,2} := S^2(T^*_x X) \otimes S^2(T^*_x X)$. Hence, $S_{2,2}$ is the tangent space of $\mathbb{A}_{j^1_x g}$ at any point.

Let d denote the flat connection of the affine space $\mathbb{A}_{j^1_x g}$.

Let $T\colon J^2 M \to \otimes^p TX$ be a morphism of bundles. The restriction of T to any of these fibres is a smooth map:

$$T_| \colon \mathbb{A}_{j^1_x g} \;\to\; \otimes^p T_x X \ ,$$

whose tangent linear map at a point $j^2_x g \in \mathbb{A}_{j^1_x g}$ is the tensor:

$$T'_{j^1_x g} := \mathrm{d}_{j^2_x g}(T) \colon S_{2,2} \longrightarrow \otimes^p T_x X \ .$$

More generally, the m^{th}-covariant derivative defines the tensor:

$$T^{m)}_{j^1_x g} := \mathrm{d}^m_{j^2_x g}(T) \colon S^m\left(S_{2,2}\right) \longrightarrow \otimes^p T_x X \ .$$

Definition. The derivative of $T\colon J^2 M \longrightarrow \otimes^p TX$ is the morphism of bundles:

$$T' \colon J^2 M \longrightarrow \otimes^p TX \otimes \left(S^2 TX \otimes S^2 TX\right) \quad , \quad j^2_x g \longmapsto T'_{j^2_x g} \ .$$

Analogously, the higher derivatives are

$$T^{m)} \colon J^2 M \longrightarrow \otimes^p TX \otimes S^m\left(S^2 TX \otimes S^2 TX\right) \quad , \quad j^2_x g \longmapsto T^{m)}_{j^2_x g} \ .$$

In local coordinates, the coefficients of T' are

$$T^{j_1 \ldots j_p;abcd} = \frac{\partial T^{j_1 \ldots j_p}}{\partial g_{ab,cd}} \ , \tag{6.1.0.1}$$

and, analogously, the coefficients of the m^{th}-derivative $T^{m)}$ are:

$$T^{j_1 \ldots j_p;a_1 b_1 c_1 d_1 \ldots a_m b_m c_m d_m} = \frac{\partial^m T^{j_1 \ldots j_p}}{\partial g_{a_1 b_1, c_1 d_1} \ldots \partial g_{a_m b_m c_m d_m}} \ .$$

The local expression of the derivative (6.1.0.1) together with Proposition 6.1.1 imply:

Corollary 6.1.2 (Lovelock)**.** *If a second-order p-tensor T is divergence-free, then its derivative*

T' satisfies the following linear symmetry:

$$0 = \sum_{(j_p\, c\, d)} T^{j_1 \dots j_p;abcd} := T^{j_1 \dots j_p;abcd} + T^{j_1 \dots d;abj_pc} + T^{j_1 \dots c;abdj_p} \; .$$

Natural tensors

Recall that a second-order tensor $T \colon J^2M \to \otimes^p TX$ is natural if it is a morphism of natural bundles.

That is, if for any diffeomorphism $\tau \colon U \to V$ between open sets of X, it holds:

$$T(\tau_* j^2_x g) = \tau_* T(j^2_x g) \; .$$

Proposition 6.1.3. *The derivative of a second-order, natural p-tensor is also a natural tensor.*

Proof: At any point $j^1_x g \in J^1M$, the naturalness of T implies the commutativity of:

$$\begin{tikzcd} \mathbb{A}_{j^1_x g} \arrow[r, "T_|"] \arrow[d, "\tau_*"'] & \otimes^p T_x X \arrow[d, "\tau_*"] \\ \mathbb{A}_{\tau_* j^1_x g} \arrow[r, "T_|"] & \otimes^p T_{\tau(x)} X \end{tikzcd} \quad .$$

Hence, the corresponding tangent linear maps at any point $j^2_x g \in \mathbb{A}_{j^1_x g}$ satisfy the commutative diagram:

$$\begin{tikzcd} S_{2,2} \arrow[r, "T'_{j^2_x g}"] \arrow[d, "\tau_*"'] & \otimes^p T_x X \arrow[d, "\tau_*"] \\ S_{2,2} \arrow[r, "T'_{\tau_*(j^2_x g)}"] & \otimes^p T_{\tau(x)} X \end{tikzcd}$$

that amounts to the naturalness of T'.

□

The space $N^2 \subset S^{2,2} := S^2(T_xX) \otimes S^2(T_xX)$ of contravariant normal tensors of order 2 is defined as the kernel of the symmetrization in the last 3 indices.

Let $j^1_x g \in J^1M$ and consider the map that sends a 2-jet to its normal tensor:

$$\pi \colon \mathbb{A}_{j^1_x g} \longrightarrow N_2 \qquad , \qquad j^2_x g \longmapsto g^2_x \; .$$

This map is affine and it can be checked that the equations of its tangent linear map π_*

are:

$$S_{2,2} \xrightarrow{\pi_*} N_2 \quad , \quad g_{ij,kl} = \frac{1}{3}\left(2S_{ij,kl} - S_{ik,jl} - S_{il,jk}\right) .$$

Moreover, it is a retract of the inclusion $N_2 \subset S_{2,2}$, and hence $S_{2,2} = N_2 \oplus \operatorname{Ker}\pi_*$.

Lemma 6.1.4. *The subspace of $S^{2,2} = S^*_{2,2}$ incident with $\operatorname{Ker}\pi_*$ is the space of contravariant normal tensors N^2.*

Proof: Taking duals in $0 \longrightarrow \operatorname{Ker}\pi_* \longrightarrow S_{2,2} \xrightarrow{\pi_*} S_{2,2}$, it follows that $\operatorname{Im}(\pi^*)$ is the incident of $\operatorname{Ker}\pi_*$:

$$0 \longleftarrow (\operatorname{Ker}\pi_*)^* \longleftarrow S^*_{2,2} \xleftarrow{\pi^*} S^*_{2,2} .$$

But $\operatorname{Im}(\pi^*) = N^2$, because the dual map $\pi^* : S^{2,2} = S^*_{2,2} \to S^*_{2,2} = S^{2,2}$ can be checked to be defined by the same formula of π_*, but for contravariant indexes.

$\square$

Proposition 6.1.5 (Symmetries of the derivative)**.** *If $T\colon J^2M \to \otimes^p TX$ is a second-order, natural p-tensor, then its derivative T' takes its values in $\otimes^p TX \otimes N^2$:*

$$\begin{tikzcd} T'\colon J^2M \arrow[r] \arrow[dr] & \otimes^p TX \otimes S^{2,2} \\ & \otimes^p TX \otimes N^2 \arrow[u, hook] \end{tikzcd} . \tag{6.1.0.2}$$

Proof: Let $j^2_x g \in J^2M$ and let $j^1_x g$ be its 1-jet, so that $j^2_x g \in \mathbb{A}_{j^1_x g}$.

As T is a natural tensor, there exists a O_{g_x}-equivariant smooth map $\mathfrak{t}\colon N_2 \longrightarrow \otimes^p T_xX$ such that

$$T\left(j^2_x \tilde{g}\right) = \mathfrak{t}\left(\tilde{g}^2_x\right)$$

for any $j^2_x \tilde{g}$ with the prefixed value g_x at x.

That is, the following triangle commutes:

$$\begin{tikzcd} \mathbb{A}_{j^1_x g} \arrow[r, "T_|"] \arrow[d, "\pi"'] & \otimes^p T_xX \\ N_2 \arrow[ur, "\mathfrak{t}"'] & \end{tikzcd} . \tag{6.1.0.3}$$

Hence, their tangent linear maps at $j^2_x g$ also commute:

$$\begin{tikzcd} S_{2,2} \arrow[r, "T'_{j^2_x g}"] \arrow[d, "\pi_*"'] & \otimes^p T_x X \\ N_2 \arrow[ur, "\mathrm{t}_*"'] & \end{tikzcd}$$

that proves $T'_{j^2_x g}$ annihilates $\operatorname{Ker}\pi_* \subset S_{2,2}$.

Therefore, via Lemma 6.1.4 and the following isomorphisms

$$\mathrm{Hom}_{\mathbb{R}}(S_{2,2}, \otimes^p T_x X) = \otimes^p T_x X \otimes (S_{2,2})^* = \otimes^p T_x X \otimes S^{2,2}$$

the map $T'(j^2_x g)$ defines an element in $\otimes^p T_x X \otimes N^2$.

□

6.2 Polynomial character of second-order, divergence-free tensors

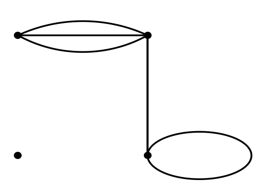
Let us begin with a short digression on graphs, that will be understood as finite CW-complexes of dimension 1:

Definition. A graph is a compact Hausdorff topological space K, together with a finite subset $K_0 \subset K$, whose elements will be called *vertices*, such that:

1. $K - K_0$ is a disjoint union of a finite collection of subspaces e_i, called *edges*, each of which is homeomorphic to an open interval.

2. The boundary of each edge is a pair of vertices or a single vertex.

Edges with equal endpoints will be called *loops*. Also, there can possibly be several edges between the same pair of vertices, v. gr:

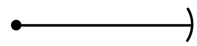


Let us denote e and v the number of edges and vertices of a graph.

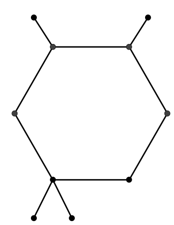
Definition. A cycle is a finite sequence of different edges $k_0 - k_1$, $k_1 - k_2, \dots, k_m - k_0$, with $k_i \neq k_j$, for $i \neq j$. A tree is a connected graph with no cycles.

Any cycle satisfies the relation $e = v$, and any tree, the relation $e = v - 1$.

Definition. A hair is a topological space homeomorphic to $[0,1)$.



A connected graph is a hairy cycle if it contains a cycle whose removal produces a disjoint union of hairs, v. gr:



In particular, any cycle is a hairy cycle (the bald case, so to speak). Any hairy cycle also satisfies the relation $e = v$.

Finally, let us say a vertex is simple if there is only one edge arriving to it (the vertex of a loop is not considered to be simple), and that a vertex is connected to a cycle if there is a cycle in its connected component.

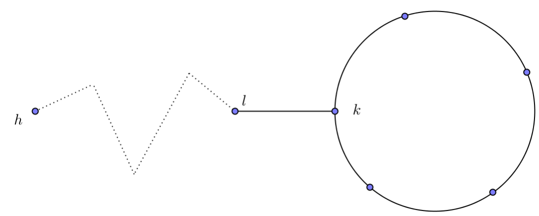
Proposition 6.2.1. *If a graph satisfies $e \geq v$, then one of the following options necessarily holds:*

1. *There exists an edge $k - l$ such that:*
    - *both k and l are not simple vertices;*
    - *after the removal of the edge $k - l$, the vertex k is still connected with a cycle.*
2. *The graph is a disjoint union of hairy cycles.*

Proof: If *1* is not satisfied, let us prove that the connected component of a cycle is a hairy cycle.

Any vertex h connected to the cycle, and not inside the cycle, has to be on an edge whose opposite endpoint is inside the cycle; otherwise there exists an edge $k - l$ satisfying *1* (see figure). Moreover, h has to be simple; otherwise, the previous edge satisfies *1*. Finally, there does not exist an edge $k-l$ between two vertices of the cycle, different from the edges of the cycle, because such an edge would satisfy *1*.

Therefore, the connected components of the graph are hairy cycles or trees.



As any hairy cycle satisfies $e = v$, and any tree $e < v-1$, the hypothesis $e \geq v$ for the graph implies no connected component is a tree.

$\square$

Vanishing of derivatives

Definition. A second-order p-tensor $T\colon J^2M \to \otimes^p TX$ is polynomial (in the second derivatives $g_{ab,cd}$) of degree $< m$ if its restrictions to the affine spaces $\mathbb{A}_{j^1_x g}$

$$T_{|}\colon \mathbb{A}_{j^1_x g} \longrightarrow \otimes^p TX$$

are polynomial maps of degree $< m$, for any fibre $\mathbb{A}_{j^1_x g}$.

This condition is equivalent to saying that the m^{th}-derivative $T^{m)}$ is null.

Definition. For $m \geq 1$, let $\mathrm{Div}^m_x \subset \otimes^p T_x X \otimes S^m N^2$ be the vector subspace whose elements satisfy:

$$0 = \sum_{(j_p\, c_1 d_1)} T^{j_1 \dots j_p\, ; a_1 b_1 c_1 d_1 \dots} .$$

Due to the symmetries of $S^m N^2$ (any quatern $a_i b_i c_i d_i$ can be put in the first position, and, by Lemma 3.2.2, $a_1 b_1$ can be interchanged with $c_1 d_1$), any element in Div^m_x satisfies:

$$\sum_{(j_p a_i b_i)} T^{j_1 \dots j_p\, ; a_1 b_1 \dots c_m d_m} = 0 = \sum_{(j_p c_i d_i)} T^{j_1 \dots j_p\, ; a_1 b_1 \dots c_m d_m} . \tag{6.2.0.4}$$

As a consequence, if the three indices $j_p a_i b_i$ are equal, then

$$T^{j_1 \dots l\, ; \dots ll \dots} = 0 .$$

Due to Corollary 6.1.2 and Theorem 6.1.5, if a natural tensor $T\colon J^2M \to \otimes^p TX$ is divergence-free, then its m^{th}- derivative takes value in this subspace:

$$\begin{tikzcd} T^{m)}\colon J^2_xM \arrow[r] \arrow[dr] & \otimes^p T_xX \otimes S^m N^2 \\ & \mathrm{Div}^m_x \arrow[u, hook] \end{tikzcd} \quad . \tag{6.2.0.5}$$

Therefore,

Proposition 6.2.2. *If there exist $m \in \mathbb{N}$ such that $\mathrm{Div}^m_x = 0$, then any divergence-free, natural p-tensor $T\colon J^2M \longrightarrow \otimes^p TX$ has to be polynomial (in the second derivatives of the metric), of degree less than m.*

Consider a component of an element in Div^m_x:

$$\mathsf{T} = T^{j_1\dots j_p;a_1b_1\dots c_md_m} \ ,$$

that we understand as a linear function on Div^m_x.

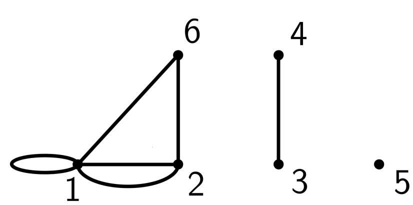
Definition. Its associated graph is defined as follows:

- It has n vertices, labelled by $i \in \{1,\dots,n\}$, where $n = \dim X$.
- For each pair of indexes a_ib_i (or c_id_i) in $T^{j_1\dots j_p;a_1b_1\dots c_md_m}$, there is one edge $a_i - b_i$ joining the vertices a_i and b_i (resp. an edge $c_i - d_i$ joining c_i and d_i).

The indexes $j_1 \dots j_p$ are irrelevant to construct the graph. As an illustration, consider the following example, in which $n = 6$ and $m = 3$:

$$\mathsf{T} = T^{j_1\dots j_p;1112;4312;6261} \ , \tag{6.2.0.6}$$

whose associated graph is:



Lemma 6.2.3. *Consider a component of an element in* Div_x^m*:*

$$\mathsf{T} = T^{j_1 \dots j_p \, ; a_1 b_1 \dots c_m d_m} \; .$$

If, in the associated graph, the vertex $j_p \in \{1, \dots, n\}$ *is connected to a cycle, then this component is zero.*

Proof: By hypothesis, there exist sequences of edges $j_p - k_1$, $k_1 - k_2$, …, $k_r - l$ connecting j_p with a cycle $l - m_1$, …, $m_s - l$.

Iterated use of (6.2.0.4) yields ($\sim$ denotes proportional by a non-zero factor):

$$\begin{aligned} T^{\dots j_p ; \dots j_p k_1 \dots} &\sim T^{\dots k_1 ; \dots j_p j_p \dots k_1 k_2 \dots} \sim T^{\dots k_2 ; \dots k_1 k_1 \dots k_2 k_3 \dots} \sim \dots \\ &\sim T^{\dots l ; \dots l m_1 \dots} \sim T^{\dots m_1 ; \dots l l \dots m_1 m_2 \dots} \sim T^{\dots m_2 ; \dots l l \dots m_2 m_3 \dots} \sim \dots \\ &\sim T^{\dots l ; \dots l l \dots} = 0 \; . \end{aligned}$$

□

Theorem 6.2.4. *If* $m \geq n/2$*, then:*

$$\mathrm{Div}_x^m = 0 \; .$$

As a consequence, any divergence-free, natural tensor $T \colon J^2 M \to \otimes^p TX$ *is polynomial (in the second derivatives of the metric), of degree less or equal than* $(n-1)/2$*.*

Proof: Let $m \geq n/2$ and suppose there exist a component T which does not vanish identically on Div_x^m.

Another component T' is considered equivalent to T if there exists $\lambda \neq 0$ such that $\mathsf{T} = \lambda \mathsf{T}'$ (as linear functions on Div_x^m) and if the last $4m+1$ indices of T' are a permutation of those in T.

Among all the elements equivalent to T, let

$$\mathsf{T}_R = T^{j_1 \dots j_p \, ; a_1 b_1 \dots c_m d_m}$$

be one with the highest number of loops (i.e., edges $k - k$ with equal endpoints) in the associated graph. As $2m \geq n$, the graph associated to T_R satisfies $e \geq v$, and we can invoke Proposition 6.2.1.

If the graph associated to T_R is a disjoint union of hairy cycles, then any vertex is connected with a cycle, and hence $\mathsf{T}_R = 0$ (Lemma 6.2.3), in contradiction with the hypothesis.

Otherwise, let $k-l$ be an edge as in *1* in Proposition 6.2.1. Then:

$$T^{\dots j_p;\dots kl\dots} = -T^{\dots k;\dots j_p l\dots} - T^{\dots l;\dots k j_p\dots} .$$

The first addend is zero because k is connected to a cycle in the graph of $T^{\dots k;\dots j_p l\dots}$.

If $l=k$, the second addend is also zero and thus $\mathsf{T}_R=0$.

In other case, $l \neq k$ and, as l is not simple, there exists at least another edge $l-m$ in the graph of $T^{\dots l;\dots k j_p\dots}$. If $m=l$, then $\mathsf{T}_R=0$, because:

$$T^{\dots l;\dots ll\dots} = 0 ,$$

and, if $m \neq l$, we also arrive to a contradiction, because we produce a component equivalent to T_R but with a greater number of loops:

$$T^{\dots j_p;\dots kl\dots} \sim T^{\dots l;\dots lm\dots} \sim T^{\dots m;\dots ll\dots} .$$

□

6.3 Computation of divergence-free tensors

Let us explain how the results in the precedent Section allow to reduce the computation of second-order p-tensors that are divergence-free to a problem of invariants for the orthogonal group.

Let g_x be a fixed metric at a point $x \in X$. By Corollary 3.3.9, any natural p-tensor $T\colon J^2M \to \otimes^p TX$ corresponds with a smooth O_{g_x}-equivariant map $\mathfrak{t}\colon N_2 \to \otimes^p T_xX$, satisfying $T(j^2_x g) = \mathfrak{t}(g^2_x)$ on each metric jet $j^2_x g$ having the prefixed value at x.

In particular, the following commutative triangle holds:

$$\begin{tikzcd} \mathbb{A}_{j^1_x g} \arrow[r, "T_{|}"] \arrow[d, "\pi"'] & \otimes^p T_xX \\ N_2 \arrow[ur, "\mathfrak{t}"'] & \end{tikzcd} ,$$

where $\pi(j^2_x g) = g^2_x$, and $j^1_x g$ is any jet with the prefixed value at x.

As π is a surjective affine map, $T_{|}$ is a homogeneous polynomial of degree m if and only if so it is $\mathfrak{t}$. Therefore, the following bijections holds:

$$\left\{ \begin{array}{l} \text{Second-order natural tensors} \quad T\colon J^2M \longrightarrow \otimes^p TX \\ \text{homogeneous of degree } m \text{ in the second derivatives} \end{array} \right\}$$

$$\|$$

$$\left\{ \begin{array}{c} O_{g_x}\text{-equivariant polynomials } \mathfrak{t}\colon N_2 \longrightarrow \otimes^p T_xX \\ \text{homogeneous of degree } m \end{array} \right\}$$

$$\|$$

$$\mathrm{Hom}_{O_{g_x}}(S^m N_2, \otimes^p T_xX) = \left(\otimes^p T_xX \otimes S^m N^2\right)^{O_{g_x}}$$

$$\|$$

$$\left\{ \begin{array}{c} \text{Zeroth-order natural tensors} \\ \tilde{T}\colon M \longrightarrow \otimes^p TX \otimes S^m N^2 \end{array} \right\}$$

We use the symbol N^2 to denote the space of contravariant normal tensors at x, as well as to denote the bundle of such tensors.

This sequence of maps sends each natural tensor $T\colon J^2M \to \otimes^p TX$, homogeneous of degree m in the second derivatives of g, to its corresponding m^{th}-derivative $T^{m)}\colon M \to \otimes^p TX \otimes S^m N^2$. Observe that, after differentiating m times, the tensor $T^{m)}$ does no longer depend on the second derivatives of the metric; hence, by naturalness, nor does it on the first derivatives.

By formula (6.2.0.5), if the p-tensor T is divergence-free, then its m^{th}-derivative (at x), $T^{m)}\colon J^2_xM \to \otimes^p T_xX \otimes S^m N^2$ takes its values inside the subspace $\mathrm{Div}^m_x \subset \otimes^p T_xX \otimes S^m N^2$.

Summing up, we have proved the following inclusion:

Theorem 6.3.1. *Let g_x be a metric at a point $x \in X$. For any $m \geq 1$ there exists an injective map*

$$\left\{ \begin{array}{l} \textit{Divergence free, natural tensors } \ T\colon J^2M \longrightarrow \otimes^p TX \\ \textit{homogeneous of degree } m \textit{ in the second derivatives} \end{array} \right\}$$

$$\cap|$$

$$\left(\mathrm{Div}^m_x\right)^{O_{g_x}}$$

that sends a tensor T to its m^{th}-derivative $T^{m)}$ at the point x.

Again, let $T\colon J^2M \longrightarrow \otimes^p TX$ be a second-order, natural p-tensor that is divergence-free. By Theorem 6.2.4, 6.2.4 T is polynomial on the second derivatives of the metric, of degree

$m \leq (n-1)/2$. Let us decompose this tensor as

$$T = T_0 + \cdots + T_m \ ,$$

where the tensors T_k are homogeneous polynomials of degree k in the second derivatives of the metric. As any diffeomorphism of the base manifold X acts linearly on the coordinates $g_{ab,cd}$ of J^2M, it is trivial to check that each tensor T_k is natural.

Let $\mathfrak{t}: N_2 \to \otimes^p T_xX$ be the O_{g_x}-equivariant polynomial corresponding to the tensor T. If $\mathfrak{t} = \mathfrak{t}_0 + \ldots + \mathfrak{t}_m$ is the decomposition into homogeneous components, then it is easy to check that each addend $\mathfrak{t}_k$ corresponds to the p-tensor T_k. This shows that the natural p-tensor T_k is homogeneous of weight $-2-2k$ respect to the metric and, consequently, so does its divergence $\operatorname{div} T_k$. In the equality

$$0 = \operatorname{div} T = \operatorname{div} T_0 + \cdots + \operatorname{div} T_m$$

each addend has a different weight, so each $\operatorname{div} T_k$ is zero.

That is, we have proved:

Proposition 6.3.2. *Any divergence free, second-order, natural p-tensor T admits a decomposition*

$$T = T_0 + \cdots + T_m \ ,$$

where each addend T_k is a divergence free, second-order, natural p-tensor which is a homogeneous polynomial of degree k in the second derivatives of the metric.

6.4 Lovelock tensors

In this Section, let us consider tensors with $p = 2$ indices.

Lemma 6.4.1. *For any $m \geq 1$:*

$$\dim \left(\mathrm{Div}_x^m\right)^{O_{g_x}} \leq 1 \ .$$

Proof: The vector space $\left(\mathrm{Div}_x^m\right)^{O_{g_x}} = \mathrm{Hom}_{O_{g_x}}\left(\mathbb{R}, \mathrm{Div}_x^m\right)$ is isomorphic to:

$$\mathrm{Hom}_{O_{g_x}}\left(\left(\mathrm{Div}_x^m\right)^*, \mathbb{R}\right)$$

which, in turn, is spanned by iterated contraction of indices (Theorem A.0.5).

Let us prove that any total contraction of indices is proportional to:

$$T^{ii;jj\ldots kk}\,,$$

where equal letters denote contraction of the corresponding positions.

We argue by descendent induction on the number of contracted pairs (i.e., contraction of an index in an odd position with the index in the following position).

Given a total contraction, if the first index is not contracted with the second one, then:

$$T^{ij;\ldots jk\ldots} \sim T^{ik;\ldots jj\ldots}$$

and the induction hypothesis applies.

Otherwise, we can assume the third index is not contracted with the fourth, and hence:

$$T^{ii;jk\ldots} = -T^{ij;ik\ldots jm\ldots} - T^{ik;ij\ldots kl\ldots} \sim T^{im;\ldots jj\ldots} + T^{il;\ldots kk\ldots}\,.$$

This two addends have the same number of contracted pairs as the original one, so we are reduced to the previous case. □

As a consequence, the vector space of divergence-free 2-tensors, homogeneous of degree $k \leq (n-1)/2$, has dimension at most one. In the following subsection, let us define explicit generators for these spaces.

Definition of the Lovelock tensors

Let g be a pseudo-Riemannian metric and let us consider it as a one-form with values on one-forms.

Its Riemann-Christoffel tensor R can also be understood as a 2-form with values on 2-forms that is symmetric, i.e., a section of $S^2\left(\Lambda^2 T^*X\right) \subset \Lambda^2 T^*X \otimes \Lambda^2 T^*X$.

With this language, the differential Bianchi identity and the torsion-free property of the Levi-Civita connection ∇ amount to the equations:

$$\mathrm{d}_\nabla R = 0 \quad , \quad \mathrm{d}_\nabla g = 0\,. \qquad (6.4.0.7)$$

With respect to the wedge product of forms, consider the following $(n-1)$-forms with values on $(n-1)$-forms:

$$\widetilde{L}_k := R \wedge \overset{k}{\ldots} \wedge R \wedge g \wedge \overset{n-2k-1}{\ldots} \wedge g$$

where k runs from 0 to the integer part of $(n-1)/2$.

These $\widetilde{L}_k$ are clearly symmetric, i.e., sections of $S^2(\Lambda^{n-1}T^*X) \subset \Lambda^{n-1}(T^*X) \otimes \Lambda^{n-1}(T^*X)$, and also satisfy $\mathrm{d}_\nabla \widetilde{L}_k = 0$, in virtue of (6.4.0.7).

The following statement is a corollary of Proposition 4.1.1:

Proposition 6.4.2. *Contraction with a volume form, ω_X, defines a linear isomorphism:*

$$TX \otimes TX \xrightarrow{\sim} \Lambda^{n-1}(T^*X) \otimes \Lambda^{n-1}(T^*X) \quad , \quad D \otimes D' \mapsto i_D\omega_X \otimes i_{D'}\omega_X \ ,$$

and symmetric 2-tensors correspond with sections of $S^2(\Lambda^{n-1}T^*X)$.

Moreover, if T and Π are a 2-tensor and a valued $(n-1)$-form corresponding via this isomorphism, then:

$$\mathrm{d}_\nabla \Pi = 0 \quad \Leftrightarrow \quad \operatorname{div} T = 0 \ .$$

Definition. The Lovelock's tensors L_k are the 2-tensors on X corresponding to the forms $\widetilde{L}_k$ via the isomorphism above.

Hence, they are symmetric 2-tensors that are divergence-free.

Examples. Apart from the trivial case of the dual metric, $L_0 = g^*$, the simplest Lovelock tensor, L_1, is proportional to the contravariant Einstein tensor; i.e., via the isomorphism above,

$$R \wedge g \wedge \overset{n-3}{\cdots} \wedge g \quad \longmapsto \quad (-1)^{q+1}(n-3)! \left(Ric - \frac{r}{2} g^* \right) \tag{6.4.0.8}$$

where q stands for the number of -1 in the signature (p,q) of g.

In particular, if X is four-dimensional and g is of signature $(+,-,-,-)$, then the valued form $R \wedge g$ exactly corresponds with the contravariant Einstein tensor $Ric - \frac{r}{2}g^*$ via the isomorphism of Proposition 6.4.2.

Let us outline how to check this formula: let $(\theta_0,\dots,\theta_n)$ be the dual basis of an oriented orthonormal basis $(D_0,\dots,D_n)$, so that

$$g = \sum_{i=1}^{n} \delta_i \theta^i \otimes \theta^i \qquad , \qquad \omega_X = \theta_0 \wedge \cdots \wedge \theta_n \ ,$$

with $\delta_i = \pm 1$ and $(-1)^q = \delta_1 \cdot \ldots \cdot \delta_n$.

As formula (6.4.0.8) is linear on R, it is enough to prove it for a basis. Let us only sketch the case $R = \delta_1\delta_2\theta_1 \wedge \theta_2 \otimes \theta_1 \wedge \theta_2$, the other elements in the basis being similar.

A direct computation shows:

$$R \wedge g \wedge \overset{n-3}{\cdots} \wedge g = \sum_{i \neq 1,2} \frac{(n-3)!(-1)^q}{\delta_i} i_{D_i}\omega_X \otimes i_{D_i}\omega_X \longmapsto \frac{(n-3)!(-1)^q}{\delta_i} \sum_{i \neq 1,2} D_i \otimes D_i \ .$$

On the other hand, $Ric = \delta_1\theta_2 \otimes \theta_1 + \delta_2\theta_2 \otimes \theta_2$ and $r = 2$, so that

$$Ric - \frac{r}{2} g^* = -\sum_{i \neq 1,2} \delta_i D_i \otimes D_i \ .$$

In general, it is easy to check that the k^{th}-Lovelock tensor L_k is a homogeneous polynomial of degree k on the second-derivatives of the metric. Therefore, L_k generates the vector space of divergence-free tensors $T \colon J^2M \to \otimes^2 TX$ that are homogeneous polynomials of degree k on the second derivatives of the metric.

To be precise:

Theorem 6.4.3. *The Lovelock tensors $L_0, \ldots, L_m$, where $2m \leq \dim X - 1$, are a basis for the $\mathbb{R}$-vector space of second-order, natural 2-tensors that are divergence-free.*

That is,

$$\left[\begin{array}{c} \textit{Natural tensors } T \colon J^2M \to \otimes^2 TX \\ \textit{that are divergence-free} \end{array} \right] = \langle L_0, \ldots, L_m \rangle \ .$$

Proof: By Theorem 6.2.4, any tensor T of the type under consideration is polynomial, of degree $m \leq (n-1)/2$ in the second derivatives of the metric.

By Proposition 6.3.2, $T = T_0 + \cdots + T_m$, where each T_k is divergence-free and is homogeneous of degree k in the second derivatives of the metric.

As the space of such tensors has dimension ≤ 1, due to Theorem 6.3.1 and Lemma 6.4.1, the tensor T_k coincides, up to a constant factor, with L_k.

□

Remarks. Our proof also characterizes the k^{th}- Lovelock tensor L_k as the only, up to a constant factor, second-order, natural 2-contravariant tensor which is divergence-free and homogeneous of weight $w = -2 - 2k$.

In ([30]), Lovelock proved a similar statement to Theorem 6.4.3, but with the additional hypothesis of symmetry on the tensors T under consideration. Later ([31]), he also established this refined version without symmetry, but only in the particular case $\dim X = 4$.

6.5 Other computations

Let $p \geq 4$ (recall there are no second-order, natural tensors with an odd number of indices).

As an illustration of the techniques explained above, let us firstly consider tensors that are symmetric in 3 indices; i.e., sections of:

$$\otimes^{p-3} T_x X \otimes S^3 T_x X \ .$$

Definition. For $m \geq 1$, let $S^3\mathrm{Div}^m_x \subset \otimes^{p-3} T_x X \otimes S^3 T_x X \otimes S^m N^2$ be the vector subspace whose elements satisfy:

$$0 = \sum_{(j_3 c_1 d_1)} T^{\dots j_1 j_2 j_3 ; a_1 b_1 c_1 d_1 \dots} \ .$$

The same computation as in Lemma 3.2.2 shows that elements in this space fulfil the symmetry:

$$T^{\dots j_1 j_2 j_3 ; a_1 b_1 c_1 d_1 \dots} = T^{\dots c_1 d_1 j_3 ; a_1 b_1 j_1 j_2 \dots} \ .$$

Proposition 6.5.1 ([8]). *For any $m \geq 1$:*

$$S^3\mathrm{Div}^m_x = 0 \ .$$

As a consequence, any divergence-free, natural tensor $T \colon J^2M \to \otimes^{p-3} TX \otimes S^3 TX$ (in one of the symmetric indices) is indeed a zeroth-order tensor.

Proof: Due to the previous symmetry:

$$3\, T^{\dots j_1 j_2 j_3 ; a_1 b_1 c_1 d_1 \dots} = T^{\dots c_1 d_1 j_3 ; a_1 b_1 j_1 j_2 \dots} + T^{\dots c_1 d_1 j_2 ; a_1 b_1 j_3 j_1 \dots} + T^{\dots c_1 d_1 j_1 ; a_1 b_1 j_2 j_3 \dots} = 0 \ .$$

□

If $p = 2k$, with $k \geq 2$, consider the following totally symmetric, natural $2k$-tensors:

$$S_{2k} := sym(g^* \otimes \overset{k}{\dots} \otimes g^*) \ .$$

As $\nabla g^* = 0$, these tensors S_{2k} are divergence-free.

Theorem 6.5.2. *If $k \geq 2$, any divergence-free, natural tensor $T \colon J^2M \to S^{2k}TX$ is a constant multiple of S_{2k}.*

Proof: Due to the previous Proposition, any totally symmetric, divergence-free tensor has to be zeroth order.

By Theorem 3.4.2, the space of zeroth-order natural tensors is isomorphic to $\mathrm{Hom}_{O_{gx}}(\mathbb{R}, S^{2k}TX) = (S^{2k}TX)^{O_{gx}}$, so the statement follows from:

$$\dim_{\mathbb{R}} \left(S^{2k}TX\right)^{O_{gx}} = 1 , \qquad \forall\, k \in \mathbb{N} ,$$

which is a trivial computation, using Theorem A.0.5.

□

Non-existence of differential forms

Definition. For $m \geq 1$ and $p > 0$, let $\Lambda\mathrm{Div}_x^m \subset \Lambda^p T_x X \otimes S^m N^2$ be the vector subspace whose elements satisfy:

$$0 = \sum_{(j_p c_1 d_1)} T^{j_1 \dots j_p ; a_1 b_1 c_1 d_1 \dots} .$$

Lemma 6.5.3. *For any $m \geq 1$:*

$$\dim_{\mathbb{R}} \left(\Lambda\mathrm{Div}_x^m\right)^{O_{gx}} = 0 .$$

Proof: The proof is similar to that of Lemma 6.4.1: let us prove that any total contraction of indices is proportional to:

$$T^{iijj\dots;kk\dots}$$

and therefore vanishes, because the contraction of two skew-symmetric indices is zero.

We argue by descendent induction on the number of contracted pairs. Given a total contraction, the first index cannot be contracted with the second one (as they are skew-symmetric), but then:

$$T^{ij\dots;\dots ik\dots} \sim T^{kj\dots;\dots ii\dots}$$

and the induction hypothesis applies.

□

Theorem 6.5.4. *There are no divergence-free natural tensors $\omega\colon J^2M \to \Lambda^p TX$, for any $p > 0$, but the zero p-vector.*

Proof: Due to the previous Lemma, any skew-symmetric tensor that is divergence-free has to be zeroth order, and the space of zeroth-order natural tensors is isomorphic to

$$\mathrm{Hom}_{O_{gx}}(\mathbb{R}, \Lambda^p TX) = (\Lambda^p TX)^{O_{gx}} = 0 ,$$

because the contraction of two skew-symmetric indices is zero.

$\square$

Appendix A

Invariant theory of the orthogonal group

Let E be a finite dimensional $\mathbb{R}$-vector space and let g be a non-singular metric of signature (p,q) on it. Let $O(p,q)$ and $SO(p,q)$ be the orthogonal and special orthogonal groups of linear isometries of (E,g), respectively.

In the last three Chapters of this memory, we make a strong use of:

Theorem A.0.5. *The vector space* $\mathrm{Hom}_{O(p,q)}(\otimes^r E, \mathbb{R})$ *of invariant linear forms on* $\otimes^r E$ *vanishes if* r *is odd, while, for* r *even, it is spanned by contractions of the type:*

$$e_1 \otimes \ldots \otimes e_r \mapsto (g \otimes \ldots \otimes g)(e_{\sigma(1)}, \ldots, e_{\sigma(r)})$$

where σ *is a permutation of* $1, \ldots, r$.

The generators of $\mathrm{Hom}_{SO(p,q)}(\otimes^r E, \mathbb{R})$ *also include linear maps of the type:*

$$e_1 \otimes \ldots \otimes e_r \mapsto (g \otimes \ldots \otimes g \otimes \omega_g \otimes \ldots \otimes \omega_g)(e_{\sigma(1)}, \ldots, e_{\sigma(r)})$$

where ω_g *is a volume form and* σ *is a permutation of* $1, \ldots, r$.

In particular, both spaces coincide if $r < n$.

This statement is well-known[1] if the orthogonal group is considered as an *affine algebraic group*; that is, if the invariance of linear forms is understood as invariance under the action of all the points of the algebraic variety $O(p,q)$.

[1] See, v.gr., Theorem 19.6 in *Grupos algebraicos y teoría de invariantes*, by C. Sancho, Sociedad Matemática Mexicana, 2001.

Nevertheless, in this memory the orthogonal group is considered as a *Lie group*; correspondingly, the invariance condition is understood as invariance under the action of the *rational(i.e., real) points* of the algebraic variety $O(p,q)$ only.

The aim of this Appendix is to prove that Theorem A.0.5 still holds in this setting. This fact has already been noticed in the literature ([10], [19]), using different arguments.

Rational points of affine $\mathbb{R}$-groups

Let $X = \operatorname{Spec} A$ be a smooth affine variety over $\mathbb{R}$; that is, A is a finitely generated $\mathbb{R}$-algebra whose local rings are regular. We assume X has not imaginary connected components.

The set of rational points, $X(\mathbb{R})$, has a canonical topology: the minimal topology for which the maps $f : X(\mathbb{R}) \to \mathbb{R}$, $f \in A$, are continuous.

Moreover, as X is smooth, $X(\mathbb{R})$ has a canonical structure of smooth manifold of the same dimension than X. This implies:

Lemma A.0.6. *The set of rational points $X(\mathbb{R})$ is Zariski dense on X.*

Proof: If X is one-dimensional, the zero-set of any algebraic function is finite; as $X(\mathbb{R})$ is a smooth manifold of dimension one, its cardinal is infinite and hence is Zariski dense.

In general, let $f \in A$ be a function vanishing on the set of rational points and let us argue on a neighbourhood of a rational point $x \in X(\mathbb{R})$. Choose non-proportional functions $h_1, h_2, \ldots \in \mathfrak{m}_x/\mathfrak{m}_x^2$ defining smooth hypersurfaces. As f vanishes on the rational points of these hypersurfaces, the induction hypothesis says $f \in (h_i)$ for any $i \in \mathbb{N}$.

As these ideals (h_i) are prime,

$$f = h_1 f_1 = h_1 h_2 f_2 = \ldots \quad \Rightarrow \quad f \in \mathfrak{m}_x^n \ \forall\, n \in \mathbb{N} \quad \Rightarrow \quad f = 0 \, .$$

□

The functor $\mathbf{Affine}_{\mathbb{R}} \to \mathbf{Lie}_{\mathbb{R}}$

Let $\mathbf{Affine}_{\mathbb{R}}$ be the category of affine $\mathbb{R}$-groups (without imaginary connected components) algebraic maps, and let $\mathbf{Lie}_{\mathbb{R}}$ be the category finite dimensional real Lie groups and smooth maps.

Affine $\mathbb{R}$-groups are smooth, so there exists a canonical functor:

$$\mathbf{Affine}_{\mathbb{R}} \to \mathbf{Lie}_{\mathbb{R}} \quad , \quad \mathrm{G} \mapsto \mathrm{G}(\mathbb{R}) \, .$$

Theorem A.0.7. *This functor* $\mathbf{Affine}_{\mathbb{R}} \to \mathbf{Lie}_{\mathbb{R}}$ *is faithful and preserves Lie algebras.*

Proof: The inclusion $\mathrm{Hom}_{alg}(G,G') \subset \mathrm{Hom}_{smth}(G(\mathbb{R}),G'(\mathbb{R}))$ follows because the set of rational points $G(\mathbb{R})$ is dense on G.

If $G = \mathrm{Spec}\, A$ is an affine group and $G(\mathbb{R})$ is its associated Lie group, let:

$$\mathfrak{g} := \mathrm{Der}_{\mathbb{R}}(A,\mathbb{R}) \quad , \quad \mathfrak{g}(\mathbb{R}) := \mathrm{Der}_{\mathbb{R}}(\mathscr{C}^{\infty}(G(\mathbb{R})),\mathbb{R})$$

be their corresponding Lie algebras.

As $G(\mathbb{R})$ is a smooth manifold whose dimension is the Krull dimension of A, both vector spaces have the same dimension, and the canonical map $\mathfrak{g}(\mathbb{R}) \hookrightarrow \mathfrak{g}$ is an isomorphism. □

Let $G = \mathrm{Spec}\, A$ be an affine $\mathbb{R}$-group; let $\mathfrak{m}_1$ be the maximal ideal of the identity element and let E be a finite dimensional $\mathbb{R}$-vector space.

Although the following definition of linear representation is not standard, it is well suited for our purposes, as it involves the ring of algebraic functions A:

Definition. A linear representation of G on E is a linear map:

$$E \xrightarrow{x\cdot} A \otimes E$$

such that the following diagrams are commutative:

$$\begin{tikzcd} E \arrow[r, "x\cdot"] \arrow[dr, "\mathrm{Id}"'] & A \otimes E \arrow[d] \\ & E \simeq A/\mathfrak{m}_1 \otimes E \end{tikzcd} \quad , \quad \begin{tikzcd} E \arrow[r, "x\cdot"] \arrow[d, "x\cdot"'] & A \otimes E \arrow[d, "m\otimes \mathrm{Id}"] \\ A \otimes E \arrow[r, "\mathrm{Id}\otimes x\cdot"] & A \otimes A \otimes E \end{tikzcd}$$

where $m\colon A \to A \otimes A$ stands for the co-product of the Hopf algebra A.

If E and $\bar{E}$ are linear representations of G, a morphism of representations between E and $\bar{E}$ is a linear map $\varphi\colon E \to \bar{E}$ such that the following square is commutative:

$$\begin{tikzcd} E \arrow[r, "x\cdot"] \arrow[d, "\varphi"'] & A \otimes E \arrow[d, "\mathrm{Id}\otimes\varphi"] \\ \bar{E} \arrow[r, "x\cdot"] & A \otimes \bar{E} \end{tikzcd} .$$

The category of linear representations of G is denoted $\mathbf{Rep}$(G).

Let $G(\mathbb{R})$ be the Lie group associated to $G = \operatorname{Spec} A$. Its ring of algebraic functions is:

$$F_{alg} := \{f : G(\mathbb{R}) \to \mathbb{R} \ , \ f \in A\} \subset \mathscr{C}^{\infty}(G(\mathbb{R})) \ .$$

Definition. Let $G = \operatorname{Spec} A$ be an affine $\mathbb{R}$-group. An algebraic linear representation of the Lie group $G(\mathbb{R})$ on a finite dimensional vector space E is a linear map:

$$E \xrightarrow{x\cdot} F_{alg} \otimes E$$

such that[2] $1 \cdot v = v$ and $(gg') \cdot v = g \cdot (g' \cdot v)$, for any rational points $g, g' \in G(\mathbb{R})$ and any vector $v \in E$.

The morphisms between algebraic linear representations of $G(\mathbb{R})$ are defined analogously as it has been done above, but replacing the algebra A by the algebra F_{alg} of algebraic functions over the rational points.

The category of algebraic linear representations of $G(\mathbb{R})$ is denoted $\mathbf{Rep_{alg}}(\mathrm{G}(\mathbb{R}))$.

There exists a canonical map $A \to F_{alg}$, that defines a "restriction" functor:

$$\mathbf{Rep(G)} \longrightarrow \mathbf{Rep_{alg}(G(\mathbb{R}))} \ .$$

Theorem A.0.8. *For any affine $\mathbb{R}$-group G without imaginary connected components, the restriction functor is an equivalence of categories:*

$$\mathbf{Rep(G)} \xrightarrow{\sim} \mathbf{Rep_{alg}(G(\mathbb{R}))} \ .$$

Proof: We have a natural identification $F_{alg} = A/I$ where I is the ideal of elements vanishing on all the rational points.

By Lemma A.0.6, the set of rational points is dense, so $I = 0$. Hence, $F_{alg} = A$ and the statement follows. $\square$.

Theorem A.0.5, in the case of the *Lie* group $O(p,q)$ and the algebraic linear representation $\otimes^k E^*$, follows as a corollary of this equivalence of categories.

[2]Here, $g \cdot v$ denotes the composition $(\delta_g \otimes \mathrm{Id}) \circ (x\cdot)(v)$, where δ_g is the evaluation map at g.

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