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Invariant Control Systems on Lie Groups: A Short Survey

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Abstract: This is a short survey of our recent research on invariant control systems (and their associated optimal control problems). We are primarily concerned with equivalence and classification, especially in three dimensions.

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1. Introduction

Geometric control theory began in the late 1960s with the study of (nonlinear) control systems by using concepts and methods from differential geometry (cf. [9, 50, 73]). A smooth control system may be viewed as a family of vector fields (or dynamical systems) on a manifold, smoothly parametrized by a set of controls. An integral curve corresponding to some admissible control function (from some time interval to the set of controls) is called a trajectory of the system. The first basic question one asks of a control system is whether or not any two points can be connected by a trajectory: this is known as the controllability problem. Once one has established that two points can be connected by a trajectory, one may wish to find a trajectory that minimizes some (practical) cost function: this is known as the optimality problem.

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A significant subclass of control systems rich in symmetry are those evolving on Lie groups and invariant under left translations; for such a system the left translation of any trajectory is a trajectory. This class of systems was first considered in 1972 by Brockett [35] and by Jurdjevic and Sussmann [53]; it forms a natural geometric framework for various (variational) problems in mathematical physics, mechanics, elasticity, and dynamical systems (cf. [9,33,50,51]). In the last few decades substantial work on applied nonlinear control has drawn attention to invariant control affine systems evolving on matrix Lie groups of low dimension (see, e.g., [52,67,69,70] and the references therein).

This paper serves as a short survey of our recent research on (the equivalence of) left-invariant control systems and the associated optimal control problems. Ideas and key results from several papers published over the last couple of years are reexamined and restructured; some elements are also reinterpreted. The first aspect we address is the equivalence of control systems. Both state space and (detached) feedback equivalence are characterized in simple algebraic terms. The classification problem in three dimensions is revisited. The second aspect we address is the equivalence of invariant optimal control problems, or rather, their associated cost-extended systems. One associates to each cost-extended system, via the Pontryagin Maximum Principle, a quadratic Hamilton–Poisson system on the associated Lie–Poisson space. Equivalence of cost-extended systems implies equivalence of the associated Hamilton–Poisson systems. Additionally, the subclass of drift-free systems with homogeneous cost are reinterpreted as invariant sub-Riemannian structures. An extended version of this survey will appear in [32].

Throughout, we make use of the classification of three-dimensional Lie groups and Lie algebras; relevant details are given in the appendix.

2. Invariant control systems and their equivalence

2.1. Invariant control affine systems. A ℓ-input left-invariant control affine system Σ on a (real, finite-dimensional, connected) Lie group G consists of a family of left-invariant vector fields Ξ_u on G, affinely parametrized by controls $u \in \mathbb{R}^\ell$. Such a system is written as

$$\dot{g} = \Xi_u(g) = \Xi(g,u) = g(A + u_1 B_1 + \cdots + u_\ell B_\ell), \qquad g \in \mathsf{G},\ u \in \mathbb{R}^\ell.$$

Here $A, B_1, \ldots, B_\ell$ are elements of the Lie algebra $\mathfrak{g}$ with $B_1, \ldots, B_\ell$ linearly independent. The "product" gA denotes the left translation $T_\mathbf{1} L_g \cdot A$ of

$A \in \mathfrak{g}$ by the tangent map of $L_g : \mathsf{G} \to \mathsf{G},\ h \mapsto gh$. (When G is a matrix Lie group, this product is simply a matrix multiplication.) Note that the dynamics $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$ are invariant under left translations, i.e., $\Xi(g,u) = g\,\Xi(\mathbf{1},u)$. Σ is completely determined by the specification of its state space G and its parametrization map $\Xi(\mathbf{1},\cdot)$. When G is fixed, we specify Σ by simply writing

$$\Sigma \,:\, A + u_1 B_1 + \cdots + u_\ell B_\ell.$$

The *trace* Γ of a system Σ is the affine subspace $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$ of $\mathfrak{g}$. (Here $\Gamma^0 = \langle B_1, \ldots, B_\ell \rangle$ is the subspace of $\mathfrak{g}$ spanned by $B_1, \ldots, B_\ell$.) A system Σ is called *homogeneous* if $A \in \Gamma^0$, and *inhomogeneous* otherwise; Σ is said to be *drift free* if $A = 0$. Also, Σ is said to have *full rank* if its trace generates the whole Lie algebra, i.e., $\mathsf{Lie}(\Gamma) = \mathfrak{g}$.

The admissible controls are piecewise continuous maps $u(\cdot) : [0,T] \to \mathbb{R}^\ell$. A *trajectory* for an admissible control $u(\cdot)$ is an absolutely continuous curve $g(\cdot) : [0,T] \to \mathsf{G}$ such that $\dot g(t) = g(t)\,\Xi(\mathbf{1}, u(t))$ for almost every $t \in [0,T]$. We say that a system Σ is *controllable* if for any $g_0, g_1 \in \mathsf{G}$, there exists a trajectory $g(\cdot) : [0,T] \to \mathsf{G}$ such that $g(0) = g_0$ and $g(T) = g_1$. If Σ is controllable, then it has full rank. For more details about invariant control systems see, e.g., [9, 50, 53, 71].

2.2. Equivalence of systems. The most natural equivalence relation for control systems is equivalence up to coordinate changes in the state space. This is called *state space equivalence* (see [47]). State space equivalence is well understood. It establishes a one-to-one correspondence between the trajectories of the equivalent systems. However, this equivalence relation is very strong. In the (general) analytic case, Krener characterized local state space equivalence in terms of the existence of a linear isomorphism preserving iterated Lie brackets of the system's vector fields ([58], see also [9, 72, 73]).

Another fundamental equivalence relation for control systems is that of *feedback equivalence*. Two feedback equivalent control systems have the same set of trajectories (up to a diffeomorphism in the state space) which are parametrized differently by admissible controls. Feedback equivalence has been extensively studied in the last few decades (see [68] and the references therein). There are a few basic methods used in the study of feedback equivalence. These methods are based either on (studying invariant properties of) associated distributions or on Cartan's method of equivalence ([41]) or inspired by the Hamiltonian formalism ([47]); also, another fruitful approach is closely related to Poincaré's technique for linearization of dynamical systems.

Feedback transformations play a crucial role in control theory, particularly in the important problem of *feedback linearization* ([48]). The study of feedback equivalence of general control systems can be reduced, by a simple trick, to the case of control affine systems ([47]). For a thorough study of the equivalence and classification of (general) control affine systems, see [39].

We consider state space equivalence and feedback equivalence in the context of left-invariant control affine systems ([29], see also [18]). Characterizations of state space equivalence and (detached) feedback equivalence are obtained in terms of Lie group isomorphisms. Furthermore, the classification of systems on the three-dimensional Lie groups is treated.

2.2.1. State space equivalence. Two systems Σ and Σ' are called *state space equivalent* if there exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that, for each control value $u \in \mathbb{R}^\ell$, the vector fields Ξ_u and Ξ'_u are ϕ-related, i.e., $T_g\phi \cdot \Xi\,(g,u) = \Xi'\,(\phi(g),u)$ for $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$. We have the following simple algebraic characterization of this equivalence.

Theorem 2.1. ([29], see also [58]) *Two full-rank systems Σ and Σ' are state space equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi \cdot \Xi\,(\mathbf{1},u) = \Xi'\,(\mathbf{1},u)$ for all $u \in \mathbb{R}^\ell$.*

Proof sketch. Suppose that Σ and Σ' are state space equivalent. By composition with a left translation, we may assume $\phi(\mathbf{1}) = \mathbf{1}$. As the elements $\Xi_u(\mathbf{1})$, $u \in \mathbb{R}^\ell$ generate $\mathfrak{g}$ and the push-forward $\phi_*\Xi_u$ of the left-invariant vector fields Ξ_u are left invariant, it follows that ϕ is a Lie group isomorphism satisfying the requisite property (cf. [18]). Conversely, suppose that $\phi : \mathsf{G} \to \mathsf{G}'$ is a Lie group isomorphism as prescribed. Then $T_g\phi \cdot \Xi(g,u) = T_{\mathbf{1}}(\phi \circ L_g) \cdot \Xi(\mathbf{1},u) = T_{\mathbf{1}}(L_{\phi(g)} \circ \phi) \cdot \Xi(\mathbf{1},u) = \Xi'(\phi(g),u)$. ∎

Remark. If ϕ is defined only between some neighbourhoods of identity of G and G', then Σ and Σ' are said to be locally state space equivalence. A characterization similar to that given in Theorem 2.1, in terms of Lie algebra automorphisms, holds ([29]). In the case of simply connected Lie groups, local and global equivalence are the same (as $d\,\mathsf{Aut}(\mathsf{G}) = \mathsf{Aut}(\mathfrak{g})$).

State space equivalence is quite a strong equivalence relation. Hence, there are so many equivalence classes that any general classification appears to be very difficult if not impossible. However, there is a chance for some reasonable classification in low dimensions. We give an example to illustrate this point.

EXAMPLE 2.1. ([1]) Any two-input inhomogeneous full-rank control affine system on the Euclidean group $\mathsf{SE}(2)$ is state space equivalent to exactly one of the following systems

$$\begin{aligned}
\Sigma_{1,\alpha\beta\gamma} &: \alpha E_3 + u_1(E_1 + \gamma_1 E_2) + u_2(\beta E_2) \\
\Sigma_{2,\alpha\beta\gamma} &: \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1(\alpha E_3) + u_2 E_2 \\
\Sigma_{3,\alpha\beta\gamma} &: \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1(E_2 + \gamma_3 E_3) + u_2(\alpha E_3).
\end{aligned}$$

Here $\alpha > 0, \beta \neq 0$ and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$, with different values of these parameters yielding distinct (non-equivalent) class representatives.

Note. A full classification (under state space equivalence) of systems on $\mathsf{SE}(2)$ appears in [1], whereas a classification of systems on $\mathsf{SE}(1,1)$ appears in [11]. For a classification of systems on $\mathsf{SO}(2,1)_0$, see [30].

2.2.2. DETACHED FEEDBACK EQUIVALENCE. We specialize feedback equivalence in the context of invariant systems by requiring that the feedback transformations are compatible with the Lie group structure (cf. [18]). Two systems Σ and Σ' are called *detached feedback equivalent* if there exist diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}'$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that, for each control value $u \in \mathbb{R}^\ell$, the vector fields Ξ_u and $\Xi'_{\varphi(u)}$ are ϕ-related, i.e., $T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \varphi(u))$ for $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$. We have the following simple algebraic characterization of this equivalence in terms of the traces $\Gamma = \operatorname{im}\Xi(\mathbf{1}, \cdot)$ and $\Gamma' = \operatorname{im}\Xi'(\mathbf{1}, \cdot)$ of Σ and Σ'.

THEOREM 2.2. ([29]) *Two full-rank systems Σ and Σ' are detached feedback equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$.*

Proof sketch. Suppose Σ and Σ' are detached feedback equivalent. By composing ϕ with an appropriate left translation, we may assume $\phi(\mathbf{1}) = \mathbf{1}'$. Hence $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}', \varphi(u))$ and so $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$. Moreover, as the elements $\Xi_u(\mathbf{1})$, $u \in \mathbb{R}^\ell$ generate $\mathfrak{g}$ and the push-forward of the left-invariant vector fields Ξ_u are left invariant, it follows that ϕ is a group isomorphism (cf. [18]). On the other hand, suppose there exists a group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$. Then there exists a unique affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^{\ell'}$ such that $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}', \varphi(u))$. As with state space equivalence, by left-invariance and the fact that ϕ is a Lie group isomorphism, it then follows that $T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \varphi(u))$. ∎

Remark. If ϕ is defined only between some neighbourhoods of identity of G and G', then Σ and Σ' are said to be locally detached feedback equivalent. A characterization similar to that given in Theorem 2.2, in terms of Lie algebra automorphisms, holds. As for state space equivalence, in the case of simply connected Lie groups local and global equivalence are the same (as $d\,\mathsf{Aut}(\mathsf{G}) = \mathsf{Aut}(\mathfrak{g})$).

Detached feedback equivalence is notably weaker than state space equivalence. To illustrate this point, we give a classification, under detached feedback equivalence, of the same class of systems considered in Example 2.1.

Example 2.2. ([23]) Any two-input inhomogeneous full-rank control affine system on $\mathsf{SE}\,(2)$ is detached feedback equivalent to exactly one of the following systems

$$\begin{aligned} \Sigma_1 &: E_1 + u_1 E_2 + u_2 E_3 \\ \Sigma_{2,\alpha} &: \alpha E_3 + u_1 E_1 + u_2 E_2. \end{aligned}$$

Here $\alpha > 0$ parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.

2.2.3. Classification in three dimensions. We exhibit a classification, under detached feedback equivalence, of the full-rank systems evolving on unimodular three-dimensional Lie groups (i.e., the classical Abelian, Heisenberg, Euclidean, semi-Euclidean, pseudo-orthogonal and orthogonal groups). We shall restrict our discussion to the simply connected groups. A representative is identified for each equivalence class. Systems on the Euclidean group and the orthogonal group are discussed as typical examples. Details on the classification of three-dimensional Lie groups and their Lie algebras (along with standard ordered bases), as well as the corresponding automorphisms groups, can be found in Appendix A.

Note. A classification, under detached feedback equivalence, of all (full-rank) control systems on three-dimensional Lie groups appears in [27] (see also [19,21–23] and [24,26]). On higher dimensional Lie groups, a classification of control systems on the orthogonal group $\mathsf{SO}\,(4)$ was obtained in [4] (see also [2]). Controllability of the respective systems is also addressed in these papers.

We start with the solvable groups; the classification procedure is as follows. Firstly, the group of automorphisms is determined (see Appendix A). Equivalence class representatives are then constructed by considering the action of an automorphism on the trace of a typical system. Lastly, one verifies that none of the representatives are equivalent.

THEOREM 2.3. ([22, 23]) *Suppose Σ is a full-rank system evolving on a simply connected unimodular solvable Lie group G. Then G is isomorphic to one of the groups listed below and Σ is detached feedback equivalent to exactly one of accompanying (full-rank) systems on that group.*

1. *On $\mathbb{R}^3$, we have the systems*

$$\Sigma^{(2,1)} : E_1 + u_1 E_2 + u_2 E_3 \qquad \Sigma^{(3,0)} : u_1 E_1 + u_2 E_2 + u_3 E_3.$$

2. *On H_3, we have the systems*

$$\Sigma^{(1,1)} : E_2 + u E_3 \qquad \Sigma^{(2,0)} : u_1 E_2 + u_2 E_3$$
$$\Sigma_1^{(2,1)} : E_1 + u_1 E_2 + u_2 E_3 \qquad \Sigma_2^{(2,1)} : E_3 + u_1 E_1 + u_2 E_2$$
$$\Sigma^{(3,0)} : u_1 E_1 + u_2 E_2 + u_3 E_3.$$

3. *On $\mathsf{SE}(1,1)$, we have the systems*

$$\Sigma_1^{(1,1)} : E_2 + u E_3 \qquad \Sigma_{2,\alpha}^{(1,1)} : \alpha E_3 + u E_2$$
$$\Sigma^{(2,0)} : u_1 E_2 + u_2 E_3 \qquad \Sigma_1^{(2,1)} : E_1 + u_1 E_2 + u_2 E_3$$
$$\Sigma_2^{(2,1)} : E_1 + u_1 (E_1 + E_2) + u_2 E_3 \qquad \Sigma_{3,\alpha}^{(2,1)} : \alpha E_3 + u_1 E_1 + u_2 E_2$$
$$\Sigma^{(3,0)} : u_1 E_1 + u_2 E_2 + u_3 E_3.$$

4. *On $\widetilde{\mathsf{SE}}(2)$, we have the systems*

$$\Sigma_1^{(1,1)} : E_2 + u E_3 \qquad \Sigma_{2,\alpha}^{(1,1)} : \alpha E_3 + u E_2$$
$$\Sigma^{(2,0)} : u_1 E_2 + u_2 E_3 \qquad \Sigma_1^{(2,1)} : E_1 + u_1 E_2 + u_2 E_3$$
$$\Sigma_{2,\alpha}^{(2,1)} : \alpha E_3 + u_1 E_1 + u_2 E_2 \qquad \Sigma^{(3,0)} : u_1 E_1 + u_2 E_2 + u_3 E_3.$$

Here $\alpha > 0$ parametrizes families of distinct (non-equivalent) class representatives.

Proof. We treat, as typical case, only item (4). The group of linearized automorphisms of $\widetilde{\mathsf{SE}}\,(2)$ is given by

$$d\,\mathsf{Aut}(\widetilde{\mathsf{SE}}\,(2)) = \left\{ \begin{bmatrix} x & y & u \\ -\sigma y & \sigma x & v \\ 0 & 0 & \sigma \end{bmatrix} \,:\, x, y, u, v \in \mathbb{R},\, x^2 + y^2 \neq 0,\, \sigma = \pm 1 \right\}.$$

Let Σ be a single-input inhomogeneous system with trace $\Gamma = A + \Gamma^0 \subset \widetilde{\mathfrak{se}}\,(2)$. Suppose $E_3^*(\Gamma^0) \neq \{0\}$. (Here E_3^* is the corresponding element of the dual basis.) Then $\Gamma = a_1E_1 + a_2E_2 + \langle b_1E_1 + b_2E_2 + E_3\rangle$. Thus

$$\psi = \begin{bmatrix} a_1 & a_1 & b_1 \\ -a_1 & a_2 & b_2 \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma_1^{(1,1)} = \Gamma$. So Σ is equivalent to $\Sigma_1^{(1,1)}$. On the other hand, suppose $E_3^*(\Gamma^0) = \{0\}$. Then $\Gamma = a_1E_1 + a_2E_2 + a_3E_3 + \langle b_1E_1 + b_2E_2\rangle$ with $a_3 \neq 0$ (as $\mathsf{Lie}(\Gamma) = \widetilde{\mathfrak{se}}(2)$). Hence

$$\psi = \begin{bmatrix} b_2\ \mathrm{sgn}(a_3) & b_1 & \frac{a_1}{a_3\ \mathrm{sgn}(a_3)} \\ -b_1\ \mathrm{sgn}(a_3) & b_2 & \frac{a_2}{a_3\ \mathrm{sgn}(a_3)} \\ 0 & 0 & \mathrm{sgn}(a_3) \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma_{2,\alpha}^{(1,1)} = \Gamma$, where $\alpha = a_3\ \mathrm{sgn}(a_3)$.

Let Σ be a two-input homogeneous system with trace $\Gamma = \langle B_1, B_2\rangle$. Then $\widehat{\Sigma}\,:\, B_1 + \langle B_2\rangle$ is a (full-rank) single-input inhomogeneous system. Therefore, there exists an automorphism ψ such that $\psi \cdot (B_1 + \langle B_2\rangle)$ equals either $E_2 + \langle E_3\rangle$ or $\alpha E_3 + \langle E_2\rangle$. Hence, in either case, we get $\psi \cdot \langle B_1, B_2\rangle = \langle E_2, E_3\rangle$. Thus Σ is equivalent to $\Sigma^{(2,0)}$.

The classification for the two-input inhomogeneous systems follows similarly. If Σ is a three-input system, then clearly it is equivalent to $\Sigma^{(3,0)}$.

Most pairs of systems cannot be equivalent due to different homogeneities or different number of inputs. As the subspace $\langle E_1, E_2\rangle$ is invariant (under the action of automorphisms), $\Sigma_1^{(1,1)}$ is not equivalent to any system $\Sigma_{2,\alpha}^{(1,1)}$. For $A \in \widetilde{\mathfrak{se}}\,(2)$ and $\psi \in d\,\mathsf{Aut}(\mathsf{SE})\,(2)$, we have that $E_3^*(\psi \cdot \alpha E_3) = \pm\alpha$. Thus $\Sigma_{2,\alpha}^{(1,1)}$ and $\Sigma_{2,\alpha'}^{(1,1)}$ are equivalent only if $\alpha = \alpha'$. For the two-input inhomogeneous systems, similar arguments hold. ∎

We now proceed to the semisimple Lie groups; the procedure for classification is similar to that of the solvable groups. However, here we employ an invariant bilinear product ω (the Lorentzian product and the dot product, respectively); the inhomogeneous systems are (partially) characterized by the level set $\{A \in \mathfrak{g} : \omega(A,A) = \alpha\}$ that their trace is tangent to.

THEOREM 2.4. ([21]) *Suppose Σ is a full-rank system evolving on a simply connected semisimple Lie group G. Then G is isomorphic to one of the groups listed below and Σ is detached feedback equivalent to exactly one of accompanying (full-rank) systems on that group.*

1. *On $\tilde{\mathsf{A}} = \widetilde{\mathsf{SL}}\,(2,\mathbb{R})$, we have the systems*

$$\begin{aligned}
&\Sigma_1^{(1,1)} : E_3 + u(E_2 + E_3) && \Sigma_{2,\alpha}^{(1,1)} : \alpha E_2 + uE_3 \\
&\Sigma_{3,\alpha}^{(1,1)} : \alpha E_1 + uE_2 && \Sigma_{4,\alpha}^{(1,1)} : \alpha E_3 + uE_2 \\
&\Sigma_1^{(2,0)} : u_1E_1 + u_2E_2 && \Sigma_2^{(2,0)} : u_1E_2 + u_2E_3 \\
&\Sigma_1^{(2,1)} : E_3 + u_1E_1 + u_2(E_2 + E_3) && \Sigma_{2,\alpha}^{(2,1)} : \alpha E_1 + u_1E_2 + u_2E_3 \\
&\Sigma_{3,\alpha}^{(2,1)} : \alpha E_3 + u_1E_1 + u_2E_2 && \Sigma^{(3,0)} : u_1E_1 + u_2E_2 + u_3E_3.
\end{aligned}$$

2. *On $\mathsf{SU}\,(2)$, we have the systems*

$$\begin{aligned}
&\Sigma_\alpha^{(1,1)} : \alpha E_2 + uE_3 && \Sigma^{(2,0)} : u_1E_2 + u_2E_3 \\
&\Sigma_\alpha^{(2,1)} : \alpha E_1 + u_1E_2 + u_2E_3 && \Sigma^{(3,0)} : u_1E_1 + u_2E_2 + u_3E_3.
\end{aligned}$$

Here $\alpha > 0$ parametrizes families of distinct (non-equivalent) class representatives.

Proof. We consider only item (2), i.e., systems on the unitary group $\mathsf{SU}\,(2)$. (The proof for item (1), although more involved, is similar.) The group of linearized automorphisms of $\mathsf{SU}\,(2)$ is $d\,\mathsf{Aut}(\mathsf{SU}\,(2)) = \mathsf{SO}\,(3) = \{g \in \mathbb{R}^{3\times 3} : gg^\top = \mathbf{1}, \det g = 1\}$. The dot product $\bullet$ on $\mathfrak{su}\,(2)$ is given by $A \bullet B = a_1b_1 + a_2b_2 + a_3b_3$. (Here $A = \sum_{i=1}^3 a_iE_i$ and $B = \sum_{i=1}^3 b_iE_i$.) The level sets $\mathcal{S}_\alpha = \{A \in \mathfrak{su}\,(2) : A \bullet A = \alpha\}$ are spheres of radius $\sqrt{\alpha}$ (and are preserved by automorphisms). The group of automorphisms acts transitively on each sphere $\mathcal{S}_\alpha$. The *critical point* $\mathfrak{C}^\bullet(\Gamma)$ (at which an inhomogeneous affine subspace is tangent to a sphere $\mathcal{S}_\alpha$) is given by

$$\mathfrak{C}^\bullet(\Gamma) = A - \frac{A \bullet B}{B \bullet B} B$$

$$\mathfrak{C}^\bullet(\Gamma) = A - \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} B_1 \bullet B_1 & B_1 \bullet B_2 \\ B_1 \bullet B_2 & B_2 \bullet B_2 \end{bmatrix}^{-1} \begin{bmatrix} A \bullet B_1 \\ A \bullet B_2 \end{bmatrix}.$$

Critical points behave well under the action of automorphisms, i.e., $\psi\cdot\mathfrak{C}^\bullet(\Gamma) = \mathfrak{C}^\bullet(\psi\cdot\Gamma)$ for any automorphism ψ. (The critical point of Γ is well defined as it is independent of parametrization.)

Let Σ be a single-input inhomogeneous system with trace Γ. There exists an automorphism ψ such that $\psi\cdot\Gamma = \alpha\sin\theta\, E_1 + \alpha\cos\theta\, E_2 + \langle E_3\rangle$, where $\alpha = \sqrt{\mathfrak{C}^\bullet(\Gamma)\bullet\mathfrak{C}^\bullet(\Gamma)}$. Hence

$$\psi' = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi'\cdot\psi\cdot\Gamma = \Gamma^{(1,1)}_\alpha$.

Let Σ be a two-input homogeneous system with trace $\Gamma = \langle B_1, B_2\rangle$. Then $\widehat{\Sigma}\,:\, B_1 + \langle B_2\rangle$ is a (full-rank) single-input inhomogeneous system. Therefore, there exists an automorphism ψ such that $\psi\cdot(B_1 + \langle B_2\rangle) = \alpha E_2 + \langle E_3\rangle$. Hence $\psi\cdot\langle B_1, B_2\rangle = \langle E_2, E_3\rangle$. Thus Σ is equivalent to $\Sigma^{(2,0)}$.

Let Σ be a two-input inhomogeneous system with trace Γ. We have $\mathfrak{C}^\bullet(\Gamma)\bullet\mathfrak{C}^\bullet(\Gamma) = \alpha^2$ for some $\alpha > 0$. As $\mathfrak{C}^\bullet(\Gamma_{1,\alpha})\bullet\mathfrak{C}^\bullet(\Gamma_{1,\alpha}) = \alpha^2$, there exists an automorphism ψ such that $\psi\cdot\mathfrak{C}^\bullet(\Gamma) = \mathfrak{C}^\bullet(\Gamma_{1,\alpha})$. Hence $\psi\cdot\Gamma$ and $\Gamma_{1,\alpha}$ are both equal to the tangent plane of $\mathcal{S}_{\alpha^2}$ at $\psi\cdot\mathfrak{C}^\bullet(\Gamma)$, and are therefore identical.

If Σ is a three-input system, then it is equivalent to $\Sigma^{(3,0)}$.

Lastly, we note that none of the representatives obtained are equivalent. (Again, we first distinguish representatives in terms of homogeneity and number of inputs.) As $\alpha^2 = \mathfrak{C}^\bullet(\Gamma^{(1,1)}_\alpha)\bullet\mathfrak{C}^\bullet(\Gamma^{(1,1)}_\alpha)$ (resp. $\alpha^2 = \mathfrak{C}^\bullet(\Gamma^{(2,1)}_\alpha)\bullet\mathfrak{C}^\bullet(\Gamma^{(2,1)}_\alpha)$) is an invariant quantity, the systems $\Sigma^{(1,1)}_\alpha$ and $\Sigma^{(1,1)}_{\alpha'}$ (resp. $\Sigma^{(2,1)}_\alpha$ and $\Sigma^{(2,1)}_{\alpha'}$) are equivalent only if $\alpha = \alpha'$. ∎

3. Invariant optimal control

We consider the class of left-invariant optimal control problems on Lie groups with fixed terminal time, affine dynamics, and affine quadratic cost. Formally, such problems are given by

$$\dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \quad g\in\mathsf{G},\ u\in\mathbb{R}^\ell \tag{1}$$

$$g(0) = g_0, \quad g(T) = g_1 \tag{2}$$

$$\mathcal{J} = \int_0^T (u(t)-\mu)^\top Q\,(u(t)-\mu)\,dt \longrightarrow \min. \tag{3}$$

Here G is a (real, finite-dimensional) connected Lie group with Lie algebra $\mathfrak{g}$, A, $B_1, \ldots, B_\ell \in \mathfrak{g}$ (with $B_1, \ldots, B_\ell$ linearly independent), $u = (u_1, \ldots, u_\ell) \in \mathbb{R}^\ell$, $\mu \in \mathbb{R}^\ell$, and Q is a positive definite $\ell \times \ell$ matrix. To each such problem, we associate a *cost-extended system* (Σ, χ). Here Σ is the control system (1) and the cost function $\chi : \mathbb{R}^\ell \to \mathbb{R}$ has the form $\chi(u) = (u - \mu)^\top Q\,(u - \mu)$. Each cost-extended system corresponds to a family of invariant optimal control problems; by specification of the boundary data (g_0, g_1, T), the associated problem is uniquely determined.

Optimal control problems of this kind have received considerable attention in the last few decades. Various physical problems have been modelled in this manner, such as optimal path planning for airplanes, motion planning for wheeled mobile robots, spacecraft attitude control, and the control of underactuated underwater vehicles ([61, 67, 75]); also, the control of quantum systems and the dynamic formation of DNA ([37,43]). Many problems (as well as sub-Riemannian structures) on various low-dimensional matrix Lie groups have been considered by a number of authors (see, e.g., [15, 16, 34, 49, 52, 54, 63, 65, 66, 69, 70]).

We introduce a form of equivalence for problems of the form (1)–(2)–(3), or rather, the associated cost-extended systems (cf. [20, 28]). *Cost equivalence* establishes a one-to-one correspondence between the associated optimal trajectories, as well as the associated extremal curves. Via the Pontryagin Maximum Principle, we associate to each cost-extended systems a quadratic Hamilton–Poisson systems on the associated Lie–Poisson space. We show that cost equivalence of cost-extended systems implies equivalence of the associated Hamiltonian systems. In addition, we reinterpret drift-free cost-extended systems (with homogeneous cost) as invariant sub-Riemannian structures.

3.1. Pontryagin Maximum Principle. The Pontryagin Maximum Principle provides necessary conditions for optimality which are naturally expressed in the language of the geometry of the cotangent bundle $T^*\mathsf{G}$ of G (see [9, 40, 50]). The cotangent bundle $T^*\mathsf{G}$ can be trivialized (from the left) such that $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$; here $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$. To an optimal control problem (1)–(2)–(3) we associate, for each real number λ and each control parameter $u \in \mathbb{R}^\ell$ a Hamiltonian function on $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$:

$$\begin{aligned} H_u^\lambda(\xi) &= \lambda\chi(u) + \xi\,(\Xi_u(g)) \\ &= \lambda\chi(u) + p\,(\Xi_u(\mathbf{1})), \qquad \xi = (g, p) \in T^*\mathsf{G}. \end{aligned} \tag{4}$$

We denote by $\vec{H}_u^\lambda$ the corresponding Hamiltonian vector field (with respect to the symplectic structure on $T^*\mathsf{G}$). In terms of the above Hamiltonians, the Maximum Principle can be stated as follows.

Maximum Principle. Suppose the controlled trajectory $(\bar{g}(\cdot), \bar{u}(\cdot))$ defined over the interval $[0, T]$ is a solution for the optimal control problem (1)–(2)–(3). Then, there exists a curve $\xi(\cdot) : [0, T] \to T^*\mathsf{G}$ with $\xi(t) \in T^*_{\bar{g}(t)}\mathsf{G}$, $t \in [0, T]$, and a real number $\lambda \leq 0$, such that the following conditions hold for almost every $t \in [0, T]$:

$$(\lambda, \xi(t)) \not\equiv (0, 0) \tag{5}$$

$$\dot{\xi}(t) = \vec{H}^\lambda_{\bar{u}(t)}(\xi(t)) \tag{6}$$

$$H^\lambda_{\bar{u}(t)}(\xi(t)) = \max_u H^\lambda_u(\xi(t)) = \text{constant}. \tag{7}$$

An optimal trajectory, $g(\cdot) : [0, T] \to \mathsf{G}$ is the projection of an integral curve $\xi(\cdot)$ of the (time-varying) Hamiltonian vector field $\vec{H}^\lambda_{\bar{u}(t)}$. A trajectory-control pair $(\xi(\cdot), u(\cdot))$ is said to be an extremal pair if $\xi(\cdot)$ satisfies the conditions (5), (6), and (7). The projection $\xi(\cdot)$ of an extremal pair is called an extremal. An extremal curve is called normal if $\lambda < 0$ and abnormal if $\lambda = 0$.

For the class of optimal control problems under consideration, the maximum condition (7) eliminates the parameter u from the family of Hamiltonians (H_u); as a result, we obtain a smooth G-invariant function H on $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$. This Hamilton–Poisson system on $T^*\mathsf{G}$ can be reduced to a Hamilton–Poisson system on the (minus) Lie–Poisson space $\mathfrak{g}^*_-$, with Poisson bracket given by

$$\{F, G\} = -p([dF(p), dG(p)]).$$

Here $F, G \in C^\infty(\mathfrak{g}^*)$ and $dF(p), dG(p)$ are elements of the double dual $\mathfrak{g}^{**}$ which is canonically identified with the Lie algebra $\mathfrak{g}$.

3.2. Equivalence of cost-extended systems. Let (Σ, χ) and (Σ', χ') be two cost-extended systems. (Σ, χ) and (Σ', χ') are said to be *cost equivalent* if there exist a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ and an affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that

$$T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u)) \qquad \text{and} \qquad \chi' \circ \varphi = r\chi$$

for $g \in \mathsf{G}$, $u \in \mathbb{R}^\ell$ and some $r > 0$. Equivalently, (Σ, χ) and (Σ', χ') are cost equivalent if and only if there exist a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ and an affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $T_\mathbf{1}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}', \varphi(u))$ and $\chi' \circ \varphi = r\chi$ for some $r > 0$. Accordingly:

- If (Σ, χ) and (Σ', χ') are cost equivalent, then Σ and Σ' are detached feedback equivalent.
- If two full-rank systems Σ and Σ' are state space equivalent, then (Σ, χ) and (Σ', χ) are cost equivalent for any cost χ.
- If two full-rank systems Σ and Σ' are detached feedback equivalent with respect to a feedback transformation φ, then $(\Sigma, \chi \circ \varphi)$ and (Σ', χ) are cost equivalent for any cost χ.

Remark. The cost-preserving condition $\chi' \circ \varphi = r\chi$ is partially motivated by the following considerations. Each cost χ on $\mathbb{R}^\ell$ induces a strict partial ordering $u < v \iff \chi(u) < \chi(v)$. It turns out that χ and χ' induce the same strict partial ordering on $\mathbb{R}^\ell$ if and only if $\chi = r\chi'$ for some $r > 0$. The dynamics-preserving condition $T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$ is just that of detached feedback equivalence (on full-rank systems).

Let $(g(\cdot), u(\cdot))$ be a controlled trajectory, defined over an interval $[0, T]$, of a cost-extended system (Σ, χ). We say that $(g(\cdot), u(\cdot))$ is a *virtually optimal controlled trajectory* (shortly VOCT) if it is a solution for the associated optimal control problem with boundary data $(g(0), g(T), T)$. Similarly, we say that $(g(\cdot), u(\cdot))$ is an *extremal controlled trajectory* (shortly ECT) if it satisfies the necessary conditions of the Pontryagin Maximum Principle (with $\lambda \leq 0$). Clearly, any VOCT is an ECT. A map $\phi \times \varphi$ defining a cost equivalence between two cost-extended systems establishes a one-to-one correspondence between their respective VOCTs (and ECTs).

Proposition 3.1. ([20, 28]) *Suppose $\phi \times \varphi$ defines a cost equivalence between (Σ, χ) and (Σ', χ'). Then*

1. *$(g(\cdot), u(\cdot))$ is a VOCT if and only if $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is a VOCT;*
2. *$(g(\cdot), u(\cdot))$ is an ECT if and only if $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is an ECT.*

One can classify the cost-extended systems corresponding to a given invariant control system by use of the following result. (We denote by $\mathsf{Aff}\,(\mathbb{R}^\ell)$ the group of affine isomorphisms of $\mathbb{R}^\ell$.)

Proposition 3.2. ([20, 28]) *Let (Σ, χ) and (Σ, χ') are two cost-extended systems (with identical underlying control system Σ) and let*

$$\mathcal{T}_\Sigma = \left\{\varphi \in \mathsf{Aff}\,(\mathbb{R}^\ell) \,:\, \exists \psi \in d\,\mathsf{Aut}(\mathsf{G}),\ \psi \cdot \Gamma = \Gamma,\ \psi \cdot \Xi(\mathbf{1}, u) = \Xi(\mathbf{1}, \varphi(u))\right\}$$

be the group of feedback transformations leaving Σ invariant. (Σ, χ) and (Σ, χ') are cost equivalent if and only if there exists an element $\varphi \in \mathcal{T}_\Sigma$ such that $\chi' = r\chi \circ \varphi$ for some $r > 0$.

Example 3.1. ([20], cf. Theorem 2.3, item 4) On $\mathsf{SE}(2)$, any full-rank two-input drift-free cost-extended system (Σ, χ) with homogeneous cost (i.e., $\Xi(\mathbf{1}, 0) = 0$ and $\chi(0) = 0$) is cost equivalent to

$$(\Sigma^{(2,0)}, \chi^{(2,0)}) \; : \; \begin{cases} \Sigma \; : \; u_1 E_2 + u_2 E_3 \\ \chi(u) = u_1^2 + u_2^2. \end{cases}$$

Example 3.2. (cf. Theorem 2.3, item 2) Any controllable cost-extended system on H_3 is C-equivalent to exactly one of the cost-extended systems

$$\left(\Sigma^{(2,0)}, \chi_1^{(2,0)}\right) : \qquad \begin{cases} \Sigma^{(2,0)} \; : \; u_1 E_2 + u_2 E_3 \\ \chi_1^{(2,0)}(u) = u_1^2 + u_2^2 \end{cases}$$

$$\left(\Sigma^{(2,0)}, \chi_2^{(2,0)}\right) : \qquad \begin{cases} \Sigma^{(2,0)} \; : \; u_1 E_2 + u_2 E_3 \\ \chi_2^{(2,0)}(u) = (u_1 - 1)^2 + u_2^2 \end{cases}$$

$$\left(\Sigma^{(2,1)}, \chi_\alpha^{(2,1)}\right) : \qquad \begin{cases} \Sigma^{(2,1)} \; : \; E_1 + u_1 E_2 + u_2 E_3 \\ \chi_\alpha^{(2,1)}(u) = (u_1 - \alpha)^2 + u_2^2 \end{cases}$$

$$\left(\Sigma^{(3,0)}, \chi_{\boldsymbol{\alpha}}^{(3,0)}\right) : \qquad \begin{cases} \Sigma^{(3,0)} \; : \; u_1 E_1 + u_2 E_2 + u_3 E_3, \\ \chi_{\boldsymbol{\alpha}}^{(3,0)}(u) = (u_1 - \alpha_1)^2 + (u_2 - \alpha_2)^2 + u_3^2. \end{cases}$$

Here $\alpha, \alpha_1, \alpha_2 \geq 0$ parametrize families of (non-equivalent) class representatives.

Note. Several examples of classification under cost-equivalence can be found in [12, 14, 17, 28]

3.3. Pontryagin lift. To any cost-extended system (Σ, χ) on a Lie group G we associate, via the Pontryagin Maximum Principle, a Hamilton–Poisson system on the associated Lie–Poisson space $\mathfrak{g}_-^*$ (cf. [9, 50, 71]). We show that equivalence of cost-extended systems implies equivalence of the associated Hamilton–Poisson systems.

Note. The Pontryagin lift may be realized as a contravariant functor between the category of cost-extended control systems and the category of Hamilton–Poisson systems ([28], see also [40]).

A *quadratic Hamilton–Poisson system* $(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$ is specified by

$$H_{A,\mathcal{Q}} : \mathfrak{g}^* \to \mathbb{R}, \quad p \mapsto p(A) + \mathcal{Q}(p).$$

Here $A \in \mathfrak{g}$ and $\mathcal{Q}$ is a quadratic form on $\mathfrak{g}^*$. If $A = 0$, then the system is called *homogeneous*; otherwise, it is called *inhomogeneous*. (When $\mathfrak{g}^*_-$ is fixed, a system $(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$ is identified with its Hamiltonian $H_{A,\mathcal{Q}}$.) To each function $H \in C^\infty(\mathfrak{g}^*)$, we associate a *Hamiltonian vector field* $\vec{H}$ on $\mathfrak{g}^*$ specified by $\vec{H}[F] = \{F, H\}$. A function $C \in C^\infty(\mathfrak{g}^*)$ is a *Casimir function* if $\{C, F\} = 0$ for all $F \in C^\infty(\mathfrak{g}^*)$, or equivalently $\vec{C} = 0$. A linear map $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$ is a *linear Poisson morphism* if $\{F, G\} \circ \psi = \{F \circ \psi, G \circ \psi\}$ for all $F, G \in C^\infty(\mathfrak{h}^*)$. Linear Poisson morphisms are exactly the dual maps of Lie algebra homomorphisms.

Let $(E_1, \ldots, E_n)$ be an ordered basis for the Lie algebra $\mathfrak{g}$ and let $(E_1^*, \ldots, E_n^*)$ denote the corresponding dual basis for $\mathfrak{g}^*$. We write elements $B \in \mathfrak{g}$ as column vectors and elements $p \in \mathfrak{g}^*$ as row vectors. Whenever convenient, linear maps will be identified with their matrices. If we write elements $u \in \mathbb{R}^\ell$ as column vectors as well, then we can express $\Xi_u(\mathbf{1}) = A + u_1 B_1 + \cdots + u_\ell B_\ell$ as $\Xi_u(\mathbf{1}) = A + \mathbf{B}\, u$, where $\mathbf{B} = \begin{bmatrix} B_1 & \cdots & B_\ell \end{bmatrix}$ is a $n \times \ell$ matrix. The equations of motion for the integral curve $p(\cdot)$ of the Hamiltonian vector field $\vec{H}$ corresponding to $H \in C^\infty(\mathfrak{g}^*)$ then take the form $\dot{p}_i = -p([E_i, dH(p)])$.

Let (Σ, χ) be a cost-extended system with

$$\Xi_u(\mathbf{1}) = A + \mathbf{B}\, u, \qquad \chi(u) = (u - \mu)^\top Q (u - \mu).$$

By the Pontryagin Maximum Principle we have the following result.

Proposition 3.3. (cf. [20, 50, 59]) *Any normal ECT $(g(\cdot), u(\cdot))$ of (Σ, χ) is given by*

$$\dot{g}(t) = \Xi(g(t), u(t)), \qquad u(t) = Q^{-1}\, \mathbf{B}^\top\, p(t)^\top + \mu$$

where $p(\cdot) : [0, T] \to \mathfrak{g}^*$ is an integral curve for the Hamilton–Poisson system on $\mathfrak{g}^*_-$ specified by

$$H(p) = p\,(A + \mathbf{B}\,\mu) + \tfrac{1}{2}\, p\, \mathbf{B}\, Q^{-1}\, \mathbf{B}^\top\, p^\top. \tag{8}$$

We say that two quadratic Hamilton–Poisson systems $(\mathfrak{g}^*_-, G)$ and $(\mathfrak{h}^*_-, H)$ are *linearly equivalent* if there exists a linear isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$ such that the Hamiltonian vector fields $\vec{G}$ and $\vec{H}$ are ψ-related, i.e., $T_p\psi \cdot \vec{G}(p) = \vec{H}(\psi(p))$ for $p \in \mathfrak{g}^*$.

PROPOSITION 3.4. *The following pairs of Hamilton–Poisson systems (on $\mathfrak{g}^*_-$, specified by their Hamiltonians) are linearly equivalent:*

1. $H_{A,Q} \circ \psi$ *and* $H_{A,Q}$*, where* $\psi : \mathfrak{g}^*_- \to \mathfrak{g}^*_-$ *is a linear Lie–Poisson automorphism;*
2. $H_{A,Q}$ *and* $H_{A,rQ}$*, where* $r > 0$*;*
3. $H_{A,Q}$ *and* $H_{A,Q} + C$*, where* C *is a Casimir function.*

THEOREM 3.1. ([28]) *If two cost-extended systems are cost equivalent, then their associated Hamilton–Poisson systems, given by* (8)*, are linearly equivalent.*

Proof. Let (Σ, χ) and (Σ', χ') be cost-extended systems with $\Xi_u(\mathbf{1}) = A + \mathbf{B}\, u$ and $\Xi'_u(\mathbf{1}) = A' + \mathbf{B}'\, u'$, respectively. The associated Hamilton–Poisson systems (on $\mathfrak{g}^*_-$ and $(\mathfrak{g}')^*_-$, respectively) are given by

$$H_{(\Sigma,\chi)}(p) = p\,(A + \mathbf{B}\,\mu) + \tfrac{1}{2}\, p\,\mathbf{B}\,Q^{-1}\,\mathbf{B}^\top\, p^\top$$
$$H_{(\Sigma',\chi')}(p) = p\,(A' + \mathbf{B}'\,\mu') + \tfrac{1}{2}\, p\,\mathbf{B}'\,Q'^{-1}\,\mathbf{B}'^\top\, p^\top.$$

Suppose $\phi \times \varphi$ defines a cost equivalence between (Σ, χ) and (Σ', χ'), where $\varphi(u) = Ru + \varphi_0$ and $R \in \mathbb{R}^{\ell \times \ell}$. We have $\chi' \circ \varphi = r\chi$ for some $r > 0$. A simple calculation yields

$$T_{\mathbf{1}}\phi \cdot A = A' + \mathbf{B}'\,\varphi_0, \quad R\mu + \varphi_0 = \mu', \quad T_{\mathbf{1}}\phi \cdot \mathbf{B} = \mathbf{B}'R, \quad R\,Q^{-1}\,R^\top = r\,(Q')^{-1}.$$

Thus $(H_{(\Sigma,\chi)} \circ (T_{\mathbf{1}}\phi)^*)(p) = p\,(A' + \mathbf{B}'\,\mu') + \frac{r}{2}\, p\,\mathbf{B}'\,(Q')^{-1}\,\mathbf{B}'^\top\, p^\top$. Here $(T_{\mathbf{1}}\phi)^* : (\mathfrak{g}')^* \to \mathfrak{g}^*$ is the dual of the linear map $T_{\mathbf{1}}\phi$. Hence, the vector fields associated with $H_{(\Sigma',\chi')}$ and $H_{(\Sigma,\chi)} \circ (T_{\mathbf{1}}\phi)^*$, respectively, are related by the dilation $\delta_{1/r} : (\mathfrak{g}')^* \to (\mathfrak{g}')^*$, $p \mapsto \frac{1}{r} p$ (Proposition 3.4). Moreover, the vector fields associated with $H_{(\Sigma,\chi)} \circ (T_{\mathbf{1}}\phi)^*$ and $H_{(\Sigma,\chi)}$, respectively, are related by the linear Poisson isomorphism $(T_{\mathbf{1}}\phi)^*$ (Proposition 3.4). Consequently $\frac{1}{r}(T_{\mathbf{1}}\phi)^*$ defines a linear equivalence between $((\mathfrak{g}')^*_-, H_{(\Sigma',\chi')})$ and $(\mathfrak{g}^*_-, H_{(\Sigma,\chi)})$. ∎

Remark. The converse of Theorem 3.1 is not true in general. In fact, one can construct cost-extended systems with different number of inputs but equivalent Hamiltonians (see, e.g., [28]).

In Example 3.2 we gave a classification of the cost extended systems on H_3. Each Hamiltonian system $((\mathfrak{h}_3)^*_-, H)$, where H is a positive definite quadratic form, can be realized as the Hamiltonian system (8) associated to some cost-extended system. Hence, by Theorem 3.1, we get the following result.

EXAMPLE 3.3. Any quadratic Hamilton–Poisson systems $((\mathfrak{h}_3)^*_-, H)$, where H is a positive definite quadratic form, is linearly equivalent to the system on $(\mathfrak{h}_3)^*_-$ with Hamiltonian $H'(p) = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2)$.

3.4. SUB-RIEMANNIAN STRUCTURES. Left-invariant sub-Riemannian (and, in particular, Riemannian) structures on Lie groups can naturally be associated to drift-free cost-extended systems with homogeneous cost. We show that if two cost-extended systems are cost equivalent, then the associated sub-Riemannian structures are isometric up to rescaling.

A *left-invariant sub-Riemannian manifold* is a triplet $(\mathsf{G}, \mathcal{D}, \mathbf{g})$, where G is a (real, finite-dimensional) connected Lie group, $\mathcal{D}$ is a nonintegrable left-invariant distribution on G, and $\mathbf{g}$ is a left-invariant Riemannian metric on $\mathcal{D}$. More precisely, $\mathcal{D}(\mathbf{1})$ is a linear subspace of the Lie algebra $\mathfrak{g}$ of G and $\mathcal{D}(g) = g\mathcal{D}(\mathbf{1})$; the metric $\mathbf{g_1}$ is a positive definite symmetric bilinear from on $\mathfrak{g}$ and $\mathbf{g}_g(gA, gB) = \mathbf{g_1}(A, B)$ for $A, B \in \mathfrak{g}$, $g \in \mathsf{G}$. When $\mathcal{D} = T\mathsf{G}$ (i.e., $\mathcal{D}(\mathbf{1}) = \mathfrak{g}$) then one has a left-invariant Riemannian structure. An absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$ is called a *horizontal curve* if $\dot{g}(t) \in \mathcal{D}(g(t))$ for almost all $t \in [0, T]$. We shall assume that $\mathcal{D}$ satisfies the bracket generating condition, i.e., $\mathcal{D}(\mathbf{1})$ has full rank; this condition is necessary and sufficient for any two points in G to be connected by a horizontal curve.

A standard argument shows that the length minimization problem

$$\dot{g}(t) \in \mathcal{D}(g(t)), \qquad g(0) = g_0, \quad g(T) = g_1,$$
$$\int_0^T \sqrt{\mathbf{g}(\dot{g}(t), \dot{g}(t))} \longrightarrow \min$$

is equivalent to the energy minimization problem, or invariant optimal control problem:

$$\begin{aligned} &\dot{g} = \Xi_u(g), \; u \in \mathbb{R}^\ell \qquad g(0) = g_0, \; g(T) = g_1 \\ &\int_0^T \chi(u(t))\, dt \longrightarrow \min. \end{aligned} \tag{9}$$

Here $\Xi_u(\mathbf{1}) = u_1 B_1 + \cdots + u_\ell B_\ell$ where $B_1, \ldots, B_\ell$ are some linearly independent elements of $\mathfrak{g}$ such that $\langle B_1, \ldots, B_\ell \rangle = \mathcal{D}(\mathbf{1})$; $\chi(u(t)) = u(t)^\top Q\, u(t) = \mathbf{g_1}(\Xi_{u(t)}(\mathbf{1}), \Xi_{u(t)}(\mathbf{1}))$ for some $\ell \times \ell$ positive definite (symmetric) matrix Q. More precisely, energy minimizers are exactly those length minimizers which have constant speed. In other words, the VOCTs of the cost-extended system (Σ, χ) associated with (9) are exactly the (constant speed) minimizing geodesics of the sub-Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$; the normal (resp. abnormal) ECTs of (Σ, χ) are the normal (resp. abnormal) geodesics of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$.

Accordingly, to a (full-rank) cost-extended system (Σ, χ) on G of the form

$$\Sigma \;:\; u_1 B_1 + \cdots + u_\ell B_\ell, \qquad \chi(u) = u^\top Q u$$

we associate a sub-Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ specified by

$$\mathcal{D}(\mathbf{1}) = \Gamma = \langle B_1, \ldots, B_\ell \rangle, \qquad \mathbf{g_1}(u_1 B_1 + \cdots + u_\ell B_\ell, u_1 B_1 + \cdots + u_\ell B_\ell) = \chi(u).$$

Let $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ be two sub-Riemannian structures associated to (Σ, χ) and (Σ', χ'), respectively.

Theorem 3.2. *(Σ, χ) and (Σ', χ') are cost equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $\phi_* \mathcal{D} = \mathcal{D}'$ and $\mathbf{g} = r\, \phi^* \mathbf{g}'$ for some $r > 0$.*

Proof. Suppose $\phi \times \varphi$ defines a cost equivalence between (Σ, χ) and (Σ', χ'), i.e., $\phi_* \Xi_u = \Xi'_{\varphi(u)}$ and $\chi' \circ \varphi = r \chi$ for some $r > 0$. As $T_{\mathbf{1}}\phi \cdot \Xi_u(\mathbf{1}) = \Xi_{\varphi(u)}(\mathbf{1})$, it follows that $T_{\mathbf{1}}\phi \cdot \mathcal{D}(\mathbf{1}) = \mathcal{D}'(\mathbf{1})$. Hence, as ϕ is a Lie group isomorphism, by left invariance we have $\phi_* \mathcal{D} = \mathcal{D}'$. Furthermore

$$\begin{aligned} & r\, \chi(u) = \chi'(\varphi(u)) \\ \iff \quad & r\, \mathbf{g_1}(\Xi_u(\mathbf{1}), \Xi_u(\mathbf{1})) = \mathbf{g}'_{\mathbf{1}}(\Xi'_{\varphi(u)}(\mathbf{1}), \Xi'_{\varphi(u)}(\mathbf{1})) \\ \iff \quad & r\, \mathbf{g_1}(\Xi_u(\mathbf{1}), \Xi_u(\mathbf{1})) = \mathbf{g}'_{\mathbf{1}}(T_{\mathbf{1}}\phi \cdot \Xi_u(\mathbf{1}), T_{\mathbf{1}}\phi \cdot \Xi_u(\mathbf{1})). \end{aligned} \tag{10}$$

Hence, as ϕ is a Lie group isomorphism, by left invariance we have $r\, \mathbf{g} = \phi^* \mathbf{g}'$.

Conversely, suppose $\phi_* \mathcal{D} = \mathcal{D}'$ and $\mathbf{g} = r\, \phi^* \mathbf{g}'$. We have $T_{\mathbf{1}}\phi \cdot \mathcal{D}(\mathbf{1}) = \mathcal{D}'(\mathbf{1})$ and so $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$. Hence there exists a unique linear map $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$

such that $T_{\mathbf{1}}\phi \cdot \Xi_u(\mathbf{1}) = \Xi'_{\varphi(u)}(\mathbf{1})$. Thus $\phi \times \varphi$ defines a detached feedback equivalence between Σ and Σ'. By (10), it follows that $\chi' \circ \varphi = r\chi$. Thus (Σ, χ) and (Σ', χ') are cost equivalent. ∎

Remark. For (sub-Riemannian) Carnot groups and invariant Riemannian structures on nilpotent Lie groups, any isometry is the composition of a left translation and a Lie group isomorphism (see [36, 45, 55] and [60, 76], respectively). Recently, this has been shown to generalize to any nilpotent metric Lie group ([56]). Hence, at least for these classes, if $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ are isometric, then (Σ, χ) and (Σ', χ') are cost equivalent.

Analogous to Example 3.1, we have the following classification of sub-Riemannian structures on the Euclidean group $\mathsf{SE}(2)$.

Example 3.4. On $\mathsf{SE}(2)$, any left-invariant sub-Riemannian structure $(\mathcal{D}, \mathbf{g})$ isometric up to rescaling to the structure $(\bar{\mathcal{D}}, \bar{\mathbf{g}})$ with orthonormal frame (E_2, E_3); here E_2 and E_3 are viewed as left-invariant vector fields.

4. Final remarks

As already mentioned, a complete classification of the invariant control affine systems in three dimensions was obtained in [27] (see also [21–23]). There is no complete classification of the cost-extended systems in three dimensions. However, there are classifications of the invariant sub-Riemannian structures ([8]) and invariant Riemannian structures ([44]). Classifications in four dimensions (and beyond) are also topics for future research.

In order to find the extremal trajectories for a cost-extended system, one needs to integrate the associated Hamilton–Poisson system (see Proposition 3.3). In the last decade or so several authors have considered quadratic Hamilton–Poisson systems on low-dimensional Lie–Poisson spaces (see, e.g., [3,6,10,16,74]). To our knowledge there is currently no general classification of the quadratic Hamilton–Poisson systems in three dimensions. A first attempt towards such a classification appears in [31] (see also [5, 7, 13, 25, 38]).

A. Three-dimensional Lie algebras and groups

There are eleven types of three-dimensional real Lie algebras; in fact, nine algebras and two parametrized infinite families of algebras (see, e.g., [57, 62, 64]). In terms of an (appropriate) ordered basis (E_1, E_2, E_3), the commutation operation is given by

$$\begin{aligned}[E_2, E_3] &= n_1 E_1 - a E_2 \\ [E_3, E_1] &= a E_1 + n_2 E_2 \\ [E_1, E_2] &= n_3 E_3.\end{aligned}$$

The structure parameters a, n_1, n_2, n_3 for each type are given in Table 1.

	a	n_1	n_2	n_3	Unimodular	Nilpotent	Compl. Solv.	Exponential	Solvable	Simple	Connected Groups
$3\mathfrak{g}_1$	0	0	0	0	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$		$\mathbb{R}^3,\ \mathbb{R}^2 \times \mathbb{T},\ \mathbb{R} \times \mathbb{T}^2,\ \mathbb{T}^3$
$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$	1	1	-1	0			$\bullet$	$\bullet$	$\bullet$		$\mathsf{Aff}\,(\mathbb{R})_0 \times \mathbb{R},\ \mathsf{Aff}\,(\mathbb{R})_0 \times \mathbb{T}$
$\mathfrak{g}_{3.1}$	0	1	0	0	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$		$\mathsf{H}_3,\ \mathsf{H}_3^*$
$\mathfrak{g}_{3.2}$	1	1	0	0			$\bullet$	$\bullet$	$\bullet$		$\mathsf{G}_{3.2}$
$\mathfrak{g}_{3.3}$	1	0	0	0			$\bullet$	$\bullet$	$\bullet$		$\mathsf{G}_{3.3}$
$\mathfrak{g}_{3.4}^0$	0	1	-1	0	$\bullet$		$\bullet$	$\bullet$	$\bullet$		$\mathsf{SE}\,(1,1)$
$\mathfrak{g}_{3.4}^a$	$a>0$, $a\neq 1$	1	-1	0			$\bullet$	$\bullet$	$\bullet$		$\mathsf{G}_{3.4}^a$
$\mathfrak{g}_{3.5}^0$	0	1	1	0	$\bullet$				$\bullet$		$\widetilde{\mathsf{SE}}\,(2),\ \mathsf{SE}_n(2),\ \mathsf{SE}\,(2)$
$\mathfrak{g}_{3.5}^a$	$a>0$	1	1	0				$\bullet$	$\bullet$		$\mathsf{G}_{3.5}^a$
$\mathfrak{g}_{3.6}$	0	1	1	-1	$\bullet$					$\bullet$	$\widetilde{\mathsf{A}},\ \mathsf{A}_n,\ \mathsf{SL}\,(2,\mathbb{R}),\ \mathsf{SO}\,(2,1)_0$
$\mathfrak{g}_{3.7}$	0	1	1	1	$\bullet$					$\bullet$	$\mathsf{SU}\,(2),\ \mathsf{SO}\,(3)$

Table 1: Three-dimensional Lie algebras

A classification of the three-dimensional (real, connected) Lie groups can be found in [42]. Let G be a three-dimensional (real, connected) Lie group with Lie algebra $\mathfrak{g}$.

1. If $\mathfrak{g}$ is Abelian, i.e., $\mathfrak{g} \cong 3\mathfrak{g}_1$, then G is isomorphic to $\mathbb{R}^3$, $\mathbb{R}^2 \times \mathbb{T}$, $\mathbb{R} \times \mathbb{T}$, or $\mathbb{T}^3$.
2. If $\mathfrak{g} \cong \mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$, then G is isomorphic to $\mathsf{Aff}\,(\mathbb{R})_0 \times \mathbb{R}$ or $\mathsf{Aff}\,(\mathbb{R})_0 \times \mathbb{T}$.
3. If $\mathfrak{g} \cong \mathfrak{g}_{3.1}$, then G is isomorphic to the Heisenberg group H_3 or the Lie group $\mathsf{H}_3^* = \mathsf{H}_3/\,\mathsf{Z}(\mathsf{H}_3(\mathbb{Z}))$, where $\mathsf{Z}(\mathsf{H}_3(\mathbb{Z}))$ is the group of integer points in the centre $\mathsf{Z}(\mathsf{H}_3) \cong \mathbb{R}$ of H_3.

4. If $\mathfrak{g} \cong \mathfrak{g}_{3.2}$, $\mathfrak{g}_{3.3}$, $\mathfrak{g}^0_{3.4}$, $\mathfrak{g}^a_{3.4}$, or $\mathfrak{g}^a_{3.5}$, then G is isomorphic to the simply connected Lie group $\mathsf{G}_{3.2}$, $\mathsf{G}_{3.3}$, $\mathsf{G}^0_{3.4} = \mathsf{SE}(1,1)$, $\mathsf{G}^a_{3.4}$, or $\mathsf{G}^a_{3.5}$, respectively. (The centres of these groups are trivial.)
5. If $\mathfrak{g} \cong \mathfrak{g}^0_{3.5}$, then G is isomorphic to the Euclidean group $\mathsf{SE}(2)$, the n-fold covering $\mathsf{SE}_n(2)$ of $\mathsf{SE}_1(2) = \mathsf{SE}(2)$, or the universal covering group $\widetilde{\mathsf{SE}}(2)$.
6. If $\mathfrak{g} \cong \mathfrak{g}_{3.6}$, then G is isomorphic to the pseudo-orthogonal group $\mathsf{SO}(2,1)_0$, the n-fold covering A_n of $\mathsf{SO}(2,1)_0$, or the universal covering group $\widetilde{\mathsf{A}}$. Here $\mathsf{A}_2 \cong \mathsf{SL}(2,\mathbb{R})$.
7. If $\mathfrak{g} \cong \mathfrak{g}_{3.7}$, then G is isomorphic to either the unitary group $\mathsf{SU}(2)$ or the orthogonal group $\mathsf{SO}(3)$.

Among these Lie groups, only H^*_3, A_n, $n \geq 3$, and $\widetilde{\mathsf{A}}$ are not matrix Lie groups.

Automorphism groups. A standard computation yields the automorphism group for each three-dimensional Lie algebra (see, e.g., [46]). With respect to the given ordered basis (E_1, E_2, E_3), the automorphism group of each solvable Lie algebra has parametrization:

$$\mathsf{Aut}(\mathfrak{g}_{3.1}) \ : \ \begin{bmatrix} yw - vz & x & u \\ 0 & y & v \\ 0 & z & w \end{bmatrix} \qquad \mathsf{Aut}(\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1) \ : \ \begin{bmatrix} x & y & u \\ y & x & v \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathsf{Aut}(\mathfrak{g}_{3.2}) \ : \ \begin{bmatrix} u & x & y \\ 0 & u & z \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathsf{Aut}(\mathfrak{g}_{3.3}) \ : \ \begin{bmatrix} x & y & z \\ u & v & w \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathsf{Aut}(\mathfrak{g}^0_{3.4}) \ : \ \begin{bmatrix} x & y & u \\ y & x & v \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} x & y & u \\ -y & -x & v \\ 0 & 0 & -1 \end{bmatrix} \qquad \mathsf{Aut}(\mathfrak{g}^a_{3.4}) \ : \ \begin{bmatrix} x & y & u \\ y & x & v \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathsf{Aut}(\mathfrak{g}^0_{3.5}) \ : \ \begin{bmatrix} x & y & u \\ -y & x & v \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} x & y & u \\ y & -x & v \\ 0 & 0 & -1 \end{bmatrix} \qquad \mathsf{Aut}(\mathfrak{g}^a_{3.5}) \ : \ \begin{bmatrix} x & y & u \\ -y & x & v \\ 0 & 0 & 1 \end{bmatrix}$$

For the semisimple Lie algebras, we have

$$\mathsf{Aut}(\mathfrak{g}_{3.6}) = \mathsf{SO}\,(2,1) = \Big\{ g \in \mathbb{R}^{3\times 3} \;:\; g^\top \operatorname{diag}(1,1,-1) g = \operatorname{diag}(1,1,-1), \; \det g = 1 \Big\}$$

$$\mathsf{Aut}(\mathfrak{g}_{3.7}) = \mathsf{SO}\,(3) = \Big\{ g \in \mathbb{R}^{3\times 3} \;:\; g g^\top = \mathbf{1}, \, \det g = 1 \Big\}.$$

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