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Moore-Penrose Inverse and Operator Inequalities

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Abstract: In this note, we shall give complete characterizations of the class of all normal operators with closed range, and the class of all selfadjoint operators with closed range multiplied by scalars in terms of some operator inequalities.

Key words: Closed range operator, Moore-Penrose inverse, selfadjoint operator, unitary operator, normal operator, operator inequality.

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1. Introduction and preliminaries results

Let $\mathfrak{B}(H)$ be the C*-algebra of all bounded linear operators acting on a complex Hilbert space H , and let $\mathcal{N}(H)$, and $\mathbb{S}(H)$ denote the class of all normal operators, and the class of all selfadjoint operators in $\mathfrak{B}(H)$, respectively.

We denote by

- $\mathfrak{I}(H)$, the group of all invertible elements in $\mathfrak{B}(H)$,
- $\mathbb{S}_0(H) = \mathbb{S}(H) \cap \mathfrak{I}(H)$, the set of all invertible selfadjoint operators in $\mathfrak{B}(H)$,
- $\mathcal{N}_0(H) = \mathcal{N}(H) \cap \mathfrak{I}(H)$, the set of all invertible normal operators in $\mathfrak{B}(H)$,
- $\mathcal{R}(H)$, the set of all operators with closed ranges in $\mathfrak{B}(H)$,
- $x \otimes y$ (where $x,\ y \in H$), the one rank operator on H defined by $(x \otimes y)\, z = \langle z, y\rangle\, x$, for every $z \in H$,
- $|S|$ the positive square root of the positive operator S^*S (where $S \in \mathfrak{B}(H)$),
- $\{S\}' = \{X \in \mathfrak{B}(H) : SX = XS\}$ the commutant of S (where $S \in \mathfrak{B}(H)$).

For $S \in \mathfrak{B}(H)$, let $R(S)$ and $\ker S$ denote the range and the kernel of S, respectively. It is known that $S \in \mathcal{R}(H)$ if and only if there exits a unique operator $S^+ \in \mathcal{R}(H)$ satisfying the following four equations

$$SS^+S = S, \quad S^+SS^+ = S^+, \quad (SS^+)^* = SS^+, \quad (S^+S)^* = S^+S.$$

Then, the operator S^+ is called the Moore-Penrose inverse of S, and it satisfies that SS^+ and S^+S are orthogonal projections onto $R(S)$ and $R(S^*)$, respectively. It is clear that if $S \in \mathfrak{I}(H)$, then $S^+ = S^{-1}$, and if $S \in \mathfrak{B}(H)$ is a surjective operator (resp. injective with closed range), then $SS^+ = I$ (rep. $S^+S = I$).

For every S in $\mathcal{R}(H)$, we associate the 2×2 matrix representation $S = \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix}$ on $R(S) \oplus \ker S^*$. The operator S is called an EP operator if $R(S^*) = R(S)$, or equivalently $S_2 = 0$ and S_1 is invertible; in this case $S^+ = \begin{bmatrix} S_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ (see [2]). Any normal operator with a closed range in $\mathfrak{B}(H)$ is an EP operator (see[4]).

One of the most essential inequalities in operator theory is the arithmetic-geometric mean inequality given by (see [1, 12])

$$\forall A, B, X \in \mathfrak{B}(H), \quad \|A^*AX + XBB^*\| \geq 2\,\|AXB\|\,. \tag{1}$$

From this inequality, we deduce immediately that for every $S \in \mathbb{S}_0(H)$ the following inequality holds

$$\forall X \in \mathfrak{B}(H), \quad \left\|SXS^{-1} + S^{-1}XS\right\| \geq 2\,\|X\|\,. \tag{2}$$

The inequality (2) was proved by Corach-Porta-Recht [6] with another motivation and independently of inequality (1). In [14], we give some characterization of some distinguished classes of operators in terms of operator inequalities. We proved that the class of all operators $S \in \mathfrak{I}(H)$ satisfying (2) is exactly the class $\mathbb{C}^*\mathbb{S}_0(H)$ (that is the class of all rotations of invertible selfadjoint operators). So, the class $\mathbb{C}^*\mathbb{S}_0(H)$ is characterized by the following property

$$\forall X \in \mathfrak{B}(H), \quad \left\|SXS^{-1} + S^{-1}XS\right\| \geq 2\,\|X\|\,, \ (S \in \mathfrak{I}(H)). \tag{3}$$

In [15], we have found two other characterizations of this last class given by

$$\forall X\in\mathfrak{B}(H),\ \left\|SXS^{-1}+S^{-1}XS\right\|=\left\|S^{*}XS^{-1}+S^{-1}XS^{*}\right\|,\ (S\in\mathfrak{I}(H)).\tag{4}$$

$$\forall X\in\mathfrak{B}(H),\ \left\|SXS^{-1}+S^{-1}XS\right\|\geq\left\|S^{*}XS^{-1}+S^{-1}XS^{*}\right\|,\ (S\in\mathfrak{I}(H)).\tag{5}$$

For the class of all invertible normal operators $\mathcal{N}_0(H)$, we have showed in [15] that this class is characterized by each of the following three properties

$$\forall X\in\mathfrak{B}(H),\ \left\|SXS^{-1}\right\|+\left\|S^{-1}XS\right\|\geq 2\left\|X\right\|,\ (S\in\mathfrak{I}(H)).\tag{6}$$

$$\begin{aligned}&\forall X\in\mathfrak{B}(H),\\ &\qquad\left\|SXS^{-1}\right\|+\left\|S^{-1}XS\right\|=\left\|S^{*}XS^{-1}\right\|+\left\|S^{-1}XS^{*}\right\|,\ (S\in\mathfrak{I}(H)).\end{aligned}\tag{7}$$

$$\begin{aligned}&\forall X\in\mathfrak{B}(H),\\ &\qquad\left\|SXS^{-1}\right\|+\left\|S^{-1}XS\right\|\geq\left\|S^{*}XS^{-1}\right\|+\left\|S^{-1}XS^{*}\right\|,\ (S\in\mathfrak{I}(H)).\end{aligned}\tag{8}$$

In this note, we consider the following extensions of the above six properties from the domain $\mathfrak{I}(H)$ to the domain $\mathcal{R}(H)$

$$\forall X\in\mathfrak{B}(H),\ \left\|SXS^{+}+S^{+}XS\right\|\geq 2\left\|SS^{+}XS^{+}S\right\|,\ (S\in\mathcal{R}(H)).\tag{9}$$

$$\forall X\in\mathfrak{B}(H),\ \left\|SXS^{+}+S^{+}XS\right\|=\left\|S^{*}XS^{+}+S^{+}XS^{*}\right\|,\ (S\in\mathcal{R}(H)).\tag{10}$$

$$\forall X\in\mathfrak{B}(H),\ \left\|SXS^{+}+S^{+}XS\right\|\geq\left\|S^{*}XS^{+}+S^{+}XS^{*}\right\|,\ (S\in\mathcal{R}(H)).\tag{11}$$

$$\forall X\in\mathfrak{B}(H),\left\|SXS^{+}\right\|+\left\|S^{+}XS\right\|\geq 2\left\|SS^{+}XS^{+}S\right\|,\ (S\in\mathcal{R}(H)).\tag{12}$$

$$\begin{aligned}&\forall X\in\mathfrak{B}(H),\\ &\qquad\left\|SXS^{+}\right\|+\left\|S^{+}XS\right\|=\left\|S^{*}XS^{+}\right\|+\left\|S^{+}XS^{*}\right\|,\ (S\in\mathcal{R}(H)).\end{aligned}\tag{13}$$

$$\begin{aligned}&\forall X\in\mathfrak{B}(H),\\ &\qquad\left\|SXS^{+}\right\|+\left\|S^{+}XS\right\|\geq\left\|S^{*}XS^{+}\right\|+\left\|S^{+}XS^{*}\right\|,\ (S\in\mathcal{R}(H)).\end{aligned}\tag{14}$$

We consider also in this note the following two properties

$$\forall X\in\mathfrak{B}(H),\ \left\|S^{2}X+XS^{2}\right\|\geq 2\left\|SXS\right\|,\ (S\in\mathcal{R}(H)).\tag{15}$$

$$\forall X\in\mathfrak{B}(H),\ \left\|S^{2}X\right\|+\left\|XS^{2}\right\|\geq 2\left\|SXS\right\|,\ (S\in\mathcal{R}(H)).\tag{16}$$

In the finite dimensional case, we showed in [13] that each of properties (9) and (15) characterizes the class $\mathbb{CS}(H)$ and each of properties (12) and (16) characterizes the class $\mathcal{N}(H)$ (here $\mathcal{R}(H) = \mathfrak{B}(H)$). In this note and in general situation, we shall show that any one of properties (9), (10), (11) and (15) characterizes the class $\mathbb{C}\left(\mathbb{S}(H) \cap \mathcal{R}(H)\right)$ and any one of properties (12), (13), (14) and (16) characterizes the class $\mathcal{N}(H) \cap \mathcal{R}(H)$.

2. Characterizations and Moore-Penrose inverse

To achieve our new characterizations, we need the following lemma.

Lemma 1. *Let $S \in \mathfrak{B}(H)$. If S is injective with a closed range (or is surjective) and satisfies property* (16)*, then it is normal.*

Proof. Assume that S is injective. It is clear that S^2 is also injective with a closed range. Hence, $S^+S = I = \left(S^2\right)^+ S^2$.

Put $P = |S|$, $Q = |S^*|$ and $R = \left|S^2\right|$. Since S is injective with a closed range, then $\ker P = \ker S = \{0\}$, and $R(P) = R(S^*S)$ is closed (since $R(S^*)$ is also closed). Thus $\ker P = \{0\}$ and $R(P) = (\ker P)^{\perp} = H$. So, P is invertible. On the other hand, since S^2 is also injective with a closed range, using the same argument as used before R is invertible.

The proof is given in six steps.

Step 1. $\{R\}' = \{P\}'$.

From (16), it follows that

$$\forall X \in \mathfrak{B}(H), \quad \|X\| + \left\|S^2X\left(S^2\right)^+\right\| \geq 2\left\|SX\left(S^2\right)^+S\right\|.$$

Since $\|SS^+\| = 1$, the last inequality shows that

$$\forall X \in \mathfrak{B}(H), \quad \|X\| + \left\|S^2X\left(S^2\right)^+\right\| \geq 2\left\|SX\left(S^2\right)^+S^2S^+\right\|.$$

Thus the following inequality holds

$$\forall X \in \mathfrak{B}(H), \quad \|X\| + \left\|S^2X\left(S^2\right)^+\right\| \geq 2\left\|SXS^+\right\|.$$

By taking the polar decomposition of each of the two operators S and S^2 in this last inequality, we obtain

$$\forall X \in \mathfrak{B}(H), \quad \|X\| + \left\|RXR^{-1}\right\| \geq 2\left\|PXP^{-1}\right\|.$$

Hence from [14, Lemma 3.2], $\{R\}^{'} \subset \{P\}^{'}$. So, from this last inequality, we obtain also the following inequality

$$\forall X \in \mathfrak{B}(H), \quad \left\|P^{-1}XP\right\| + \left\|\left(RP^{-1}\right) X \left(RP^{-1}\right)^{-1}\right\| \geq 2\left\|X\right\|.$$

where RP^{-1} is a positive invertible operator. So, from [14, Theorem 3.3], $\{P\}^{'} = \left\{RP^{-1}\right\}^{'}$. Hence $\{R\}^{'} = \{P\}^{'}$.

Step 2. $\left(S^2\right)^+ S = S^+$.

From the property (16) the following inequality holds

$$\forall X \in \mathfrak{B}(H), \quad \|XS\| + \left\|S^2XS^+\right\| \geq 2\left\|SX\right\|. \tag{$*$}$$

It is known that S^+ is the unique solution of the following four equations: $SXS = S$, $XSX = X$, $(XS)^* = XS$, $(SX)^* = SX$. It is easy to see that $\left(S^2\right)^+ S$ satisfies the first three equations.

Now we prove that $\left(S^2\right)^+ S$ also satisfies the last equation. Since the operator $S\left(S^2\right)^+ S$ is a projection, it suffices to prove that its norm is less than or equal to one. By taking $X = \left(S^2\right)^+ S$ in $(*)$, we obtain

$$2 \geq \left\|\left(S^2\right)^+ S^2\right\| + \left\|S^2\left(S^2\right)^+ SS^+\right\| \geq 2\left\|S\left(S^2\right)^+ S\right\|.$$

Hence $\left\|S\left(S^2\right)^+ S\right\| \leq 1$. Therefore $\left(S^2\right)^+ S = S^+$.

Step 3. $\left(S^2\right)^+ = (S^+)^2$.

Since $S^2\left(S^2\right)^+ = SS^+S^2\left(S^2\right)^+$, then $S^2\left(S^2\right)^+ = S^2\left(S^2\right)^+ SS^+$. So from Step 2, we obtain $S^2\left(S^2\right)^+ = S^2(S^+)^2$. Since S^2 is injective, we have $\left(S^2\right)^+ = (S^+)^2$.

Step 4. P and R are 2×2 diagonal matrices with respect to the orthogonal direct sum $H = R(S) \oplus \ker S^*$.

All matrices given here are given with respect to the orthogonal direct sum $H = R(S) \oplus \ker S^*$. Put $S = \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix}$. Then $S^2 = \begin{bmatrix} S_1^2 & S_1S_2 \\ 0 & 0 \end{bmatrix}$. Since $\left(S^2\right)^+ = (S^+)^2$, then the operators S^*S and SS^+ commute (see [3, 10]). Hence $P^2 = \begin{bmatrix} S_1^*S_1 & 0 \\ 0 & S_2^*S_2 \end{bmatrix}$. So that $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$, where $P_1 = |S_1|$ and $P_2 = |S_2|$. From Step 1 and $SS^+ \in \{S^*S\}^{'}$, we deduce that $\left(S^2\right)^* S^2$ and SS^+

commute. Therefore $R^2 = \begin{bmatrix} (S_1^2)^* S_1^2 & 0 \\ 0 & (S_2^2)^* S_2^2 \end{bmatrix}$. Hence $R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$, where $R_1 = |S_1^2|$ and $R_2 = |S_2^2|$.

Step 5. $\ker S^* = \{0\}$.

Since S is injective, $\ker S^* = \{0\}$ if and only if $S_2 = 0$.

Assume that $\ker S^* \neq \{0\}$. It is easy to see that $SS^* = \begin{bmatrix} S_1S_1^* + S_2S_2^* & 0 \\ 0 & 0 \end{bmatrix}$.

Then $Q = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix}$, where $Q_1 = (S_1S_1^* + S_2S_2^*)$.

Using the polar decomposition of the operators S, S^* and S^2 in $(*)$ we obtain the following inequality

$$\forall X \in \mathfrak{B}(H), \quad \|XQ\| + \left\|RXP^{-1}\right\| \geq 2\,\|PX\|. \tag{$**$}$$

By putting $X = 0 \oplus YP_2$ (where $Y \in \mathfrak{B}(\ker S^*)$) in inequality $(**)$, we obtain

$$\forall\, Y \in \mathfrak{B}(\ker S^*), \quad \|R_2Y\| \geq 2\,\|P_2YP_2\|.$$

Hence $\left\|S_2^2\right\| = \|R_2\| \geq 2\,\|P_2\|^2 = 2\,\|S_2\|^2$. So that $\|S_2\|^2 \geq 2\,\|S_2\|^2$. Thus $S_2 = 0$, which is a contradiction with $\ker S^* \neq \{0\}$.

Therefore $\ker S^* = \{0\}$.

Step 6. S is normal.

From Step 5, S is surjective. So that S is invertible and satisfies property (16). Thus S satisfies property (6). Hence S is normal.

With the second assumption “S surjective”, S^* is injective with a closed range satisfying also property (16), so that S^* is normal. Hence S is normal. ∎

Theorem 1. *Let $S \in \mathcal{R}(H)$. Then the following properties are equivalent:*

(i) $S \in \mathcal{N}(H)$,

(ii) $\forall X \in \mathfrak{B}(H),\ \|SXS^+\| + \|S^+XS\| = \|S^*XS^+\| + \|S^+XS^*\|$,

(iii) $\forall X \in \mathfrak{B}(H),\ \|SXS^+\| + \|S^+XS\| \geq \|S^*XS^+\| + \|S^+XS^*\|$,

(iv) $\forall X \in \mathfrak{B}(H),\ \|SXS^+\| + \|S^+XS\| \geq 2\,\|SS^+XS^+S\|$,

(v) $\forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\| + \left\|XS^2\right\| \geq 2\,\|SXS\|$.

Proof. The proof is trivial if $S = 0$. Assume now that $S \neq 0$.

(i)⇒(ii). Assume $S \in \mathcal{N}(H)$. Then the equality $\|SX\| = \|S^*X\|$ holds for every $X \in \mathfrak{B}(H)$. Hence the equalities $\|SXS^+\| = \|S^*XS^+\|$ and $\|S^+XS\| = \|S^+XS^*\|$ hold for every $X \in \mathfrak{B}(H)$. So, we obtain (ii).

The implication (ii)⇒(iii) is trivial.

(iii)⇒(vi). This implication follows immediately using [11, Theorem 2.4].

(iv)⇒(v). Assume (iv) holds. Then the following inequality holds

$$\forall X \in \mathfrak{B}(H), \quad \left\|S^2XSS^+\right\| + \left\|S^+SXS^2\right\| \geq 2\left\|SS^+SXSS^+S\right\|.$$

From this inequality and since $\|SS^+\| = \|S^+S\| = 1$, property (v) follows immediately.

(v)⇒(i). Assume (v) holds. Let $S = \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R(S) \\ \ker S^* \end{bmatrix}$ and let $S^* = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R(S^*) \\ \ker S \end{bmatrix}$. Put $X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R(S) \\ \ker S^* \end{bmatrix}$. By a simple computation, we obtain

$$S^2X = \begin{bmatrix} S_1^2X_1 & 0 \\ 0 & 0 \end{bmatrix}, \; XS^2 = \begin{bmatrix} X_1S_1^2 & X_1S_1S_2 \\ 0 & 0 \end{bmatrix}, \; SXS = \begin{bmatrix} S_1X_1S_1 & S_1X_1S_2 \\ 0 & 0 \end{bmatrix}.$$

Put $Y = \begin{bmatrix} Y_1 & Y_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R(S) \\ \ker S^* \end{bmatrix}$, where Y denotes one of the above three operators. Then $\|Y\|^2 = \|YY^*\| = \left\| \begin{bmatrix} Y_1Y_1^* + Y_2Y_2^* & 0 \\ 0 & 0 \end{bmatrix} \right\| = \|Y_1Y_1^* + Y_2Y_2^*\|$. Hence we have $\left\|S^2X\right\| = \left\|S_1^2X_1\right\|$,

$$\begin{aligned} \left\|XS^2\right\|^2 &= \left\|(X_1S_1^2)(X_1S_1^2)^* + (X_1S_1S_2)(X_1S_1S_2)^*\right\| \\ &= \left\|X_1S_1K^2(X_1S_1)^*\right\| = \|X_1S_1K\|^2, \end{aligned}$$

and

$$\begin{aligned} \|SXS\| &= \|(S_1X_1S_1)(S_1X_1S_1)^* + (S_1X_1S_2)(S_1X_1S_2)^*\| \\ &= \left\|(S_1X_1)K^2(S_1X_1)^*\right\| = \|S_1X_1K\|^2, \end{aligned}$$

(where K is the positive square root of the positive operator $S_1S_1^* + S_2S_2^*$). Then using (v), we obtain the following inequality

$$\forall X_1 \in \mathfrak{B}(R(S)), \quad \left\|S_1^2X_1\right\| + \|X_1S_1K\| \geq 2\|S_1X_1K\| \tag{*}$$

On the other hand, if we put $X = x \otimes y$ (for $x,\ y \in H$) in (v), we obtain

$$\forall x, y \in H, \quad \|y\|\left\|S^2x\right\| + \|x\|\left\|(S^*)^2y\right\| \geq 2\|Sx\|\|S^*y\| \tag{**}$$

We shall prove (i) in three steps.

STEP 1. S_1 or T_1 is bounded below.

Assume that it is not the case. With the condition "S_1 is not bounded below", we may choose a sequence (u_n) in H such that

$$S^2 u_n \to 0 \text{ and } \|Su_n\| = 1, \text{ for } n \geq 1.$$

For every $n \geq 1$, there exist $x_n \in R(S^*)$ and $z_n \in \ker S$ such that $u_n = x_n + z_n$. Thus, we obtain

$$S^2 x_n = S^2 u_n \to 0,\ \|Sx_n\| = \|Su_n\| = 1,\ \|x_n\| = \left\|S^+ S u_n\right\| \leq \left\|S^+\right\|, \text{ for } n \geq 1$$

With the second condition "T_1 is not bounded below", by the same argument, we may choose a bounded sequence (y_n) in H satisfying

$$(S^*)^2 y_n \to 0,\ \|S^* y_n\| = 1, \text{ for } n \geq 1.$$

Applying (**) for $x = x_n$ and $y = y_n$, we obtain

$$\forall n \geq 1, \quad \|y_n\| \left\|S^2 x_n\right\| + \|x_n\| \left\|(S^*)^2 y_n\right\| \geq 2.$$

Letting $n \to \infty$, we have $0 \geq 2$, which is impossible. Therefore S_1 or T_1 is bounded below.

STEP 2. S_1 or T_1 is surjective.

Assume that T_1 is bounded below. Then there exists a constant $k > 0$ such that

$$\forall x \in H, \quad \left\|(S^*)^2 x\right\| \geq k \|S^* x\|.$$

So we have $S^2(S^*)^2 \geq k^2 SS^*$. From [7], we obtain $R(S^2) \supset R(S)$. Thus $R(S^2) = R(S)$. So S_1 is surjective.

Also, if S_1 is bounded below, then by the same argument, we deduce that T_1 is surjective.

STEP 3. S is normal.

Assume that S_1 is surjective (on $R(S)$). Then S_1^2 is also surjective on $R(S)$. Since $R(S) \neq \{0\}$, thus $S_1 S_1^+ = I_1 = S_1^2 \left(S_1^2\right)^+$ (where I_1 is the identity operator on $R(S)$), $S_1^+ S_1$ and $\left(S_1^2\right)^+ S_1^2$ are nonzero orthogonal projections. By putting $X_1 = \left(S_1^2\right)^+$ in (*), we obtain

$$\left\|S_1^2 \left(S_1^2\right)^+\right\| + \left\|\left(S_1^2\right)^+ S_1 K\right\| \geq 2 \left\|S_1 \left(S_1^2\right)^+ K\right\|$$

Hence, $\left\|S_1^2(S_1^2)^+\right\| = 1$, $\left\|(S_1^2)^+S_1K\right\| = \left\|(S_1^2)^+S_1^2S_1^+K\right\| \leq \left\|S_1^+K\right\|$, and $\left\|S_1(S_1^2)^+K\right\| \geq \left\|S_1^+S_1^2(S_1^2)^+K\right\| = \left\|S_1^+K\right\|$. Thus $1 \geq \left\|S_1^+K\right\|$. Hence

$$1 \geq \left\|S_1^+K\right\|^2 = \left\|S_1^+K^2(S_1^+)^*\right\| = \left\|S_1^+S_1 + (S_1^+S_2)(S_1^+S_2)^*\right\| \geq \left\|S_1^+S_1\right\| = 1.$$

Hence $\left\|S_1^+S_1 + (S_1^+S_2)(S_1^+S_2)^*\right\| = 1$. Since $S_1^+S_1$ is an orthogonal projection, by a simple computation, we obtain that $S_1^+S_1S_1^+S_2 = 0$. Hence $S_2 = S_1S_1^+S_1S_1^+S_2 = 0$. So, we obtain that S_1 is a surjective operator (as element in $\mathfrak{B}\left(R(S)\right)$ and satisfies the following inequality

$$\forall X \in \mathfrak{B}(R(S)), \quad \left\|S_1^2X\right\| + \left\|XS_1^2\right\| \geq 2\left\|S_1XS_1\right\|$$

Utilizing Lemma 1, we obtain that S_1 is normal. Hence S is normal.

With the second assumption "T_1 surjective", and since S^* satisfies (v), by using the same argument as used with the first assumption, we obtain also that S^* is normal. Thus S is normal. ∎

Corollary 1. *Assume* $\dim H < \infty$. *The class* $\mathcal{N}(H)$ *is characterized by each of the following properties*

$\forall X \in \mathfrak{B}(H), \left\| SXS^+\right\| + \left\|S^+XS\right\| \geq 2\left\|SS^+XS^+S\right\|, \ (S \in \mathfrak{B}(H)),$

$\forall X \in \mathfrak{B}(H), \ \left\|SXS^+\right\| + \left\|S^+XS\right\| = \left\|S^*XS^+\right\| + \left\|S^+XS^*\right\|, \ (S \in \mathfrak{B}(H)),$

$\forall X \in \mathfrak{B}(H), \ \left\|SXS^+\right\| + \left\|S^+XS\right\| \geq \left\|S^*XS^+\right\| + \left\|S^+XS^*\right\|, \ (S \in \mathfrak{B}(H)),$

$\forall X \in \mathfrak{B}(H), \ \left\|S^2X\right\| + \left\|XS^2\right\| \geq 2\left\|SXS\right\|, \ (S \in \mathfrak{B}(H)).$

Theorem 2. *Let* $S \in \mathcal{R}(H)$. *Then the following properties are equivalent:*

(i) $S \in \mathbb{CS}(H)$,

(ii) $\forall X \in \mathfrak{B}(H), \ \left\|SXS^+ + S^+XS\right\| = \left\|S^*XS^+ + S^+XS^*\right\|$,

(iii) $\forall X \in \mathfrak{B}(H), \ \left\|SXS^+ + S^+XS\right\| \geq \left\|S^*XS^+ + S^+XS^*\right\|$,

(iv) $\forall X \in \mathfrak{B}(H), \ \left\|SXS^+ + S^+XS\right\| \geq 2\left\|SS^+XS^+S\right\|$,

(v) $\forall X \in \mathfrak{B}(H), \ \left\|S^2X + XS^2\right\| \geq 2\left\|SXS\right\|$.

Proof. The implications (i)⇒(ii) and (ii)⇒(iii) are trivial. The implication (iii)⇒(iv) follows immediately from [11, Theorem 2.4]. The implication (i)⇒(v) follows immediately from (1).

Assume now that (iv) or (v) holds. Applying the triangular inequality in (iv) or (v), we obtain from Theorem 1, that S is normal (with a closed range).

So that S is an EP operator satisfying (iv) or (v). So $S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R(S) \\ \ker S^* \end{bmatrix}$ where S_1 is invertible on $R(S)$. Hence we obtain the following inequality

$$\forall X \in \mathfrak{B}(R(S)), \quad \left\| S_1 X S_1^{-1} + S_1^{-1} X S_1 \right\| \geq 2 \left\| X \right\|.$$

Hence S_1 is a selfadjoint operator in $\mathfrak{B}(R(S))$ multiplied by a nonzero scalar. Thus $S \in \mathbb{CS}(H)$. ∎

Corollary 2. *Assume that* $\dim H < \infty$. *The class* $\mathbb{CS}(H)$ *is characterized by each of the four following properties*

$\forall X \in \mathfrak{B}(H),\ \|SXS^+ + S^+XS\| \geq 2\,\|SS^+XS^+S\|,\ (S \in \mathfrak{B}(H)),$

$\forall X \in \mathfrak{B}(H),\ \|SXS^+ + S^+XS\| = \|S^*XS^+ + S^+XS^*\|,\ (S \in \mathfrak{B}(H)),$

$\forall X \in \mathfrak{B}(H),\ \|SXS^+ + S^+XS\| \geq \|S^*XS^+ + S^+XS^*\|,\ (S \in \mathfrak{B}(H)),$

$\forall X \in \mathfrak{B}(H),\ \left\|S^2X + XS^2\right\| \geq 2\,\|SXS\|,\ (S \in \mathfrak{B}(H)).$

Remarks. 1. Let the following extension of the property (15) to the domain $\mathfrak{B}(H)$:

$$\forall X \in \mathfrak{B}(H), \quad \left\|S^2X + XS^2\right\| \geq 2\,\|SXS\|,\ (S \in \mathfrak{B}(H)). \tag{17}$$

The restriction of this property to the domain $\mathcal{R}(H)$ characterizes the class $\mathbb{C}(\mathbb{S}(H) \cap \mathcal{R}(H))$ (by using Theorem 2). On the other hand, by using the inequality (1), the property (17) is satisfied for every $S \in \mathbb{CS}(H)$. So, does property (17) characterize the class $\mathbb{CS}(H)$?

2. Let the following extension of the property (16) to the domain $\mathfrak{B}(H)$:

$$\forall X \in \mathfrak{B}(H), \quad \left\|S^2X\right\| + \left\|XS^2\right\| \geq 2\,\|SXS\|,\ (S \in \mathfrak{B}(H)). \tag{18}$$

The restriction of this property to the domain $\mathcal{R}(H)$ characterizes the class $\mathcal{N}(H) \cap \mathcal{R}(H)$ (by using Theorem 1). The property (18) is satisfied for every $S \in \mathcal{N}(H)$. Indeed, if $S \in \mathcal{N}(H)$, then $\left\|S^2X\right\| + \left\|XS^2\right\| = \|S^*SX\| + \|XSS^*\| \geq \|S^*SX + XSS^*\| \geq 2\,\|SXS\|$, for every $X \in \mathfrak{B}(H)$. So, does property (18) characterize the class $\mathcal{N}(H)$?

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