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Subspaces of Real Four-Dimensional Lie Algebras: a Classification of Subalgebras, Ideals, and Full-Rank Subspaces

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Abstract: We classify the subspaces of each real four-dimensional Lie algebra, up to automorphism. Enumerations of the subalgebras, ideals, and full-rank (or bracket generating) subspaces are obtained. Also, the interplay between quotients (resp. extensions) of algebras and such classifications is briefly considered.

Key words: Lie algebra, subspace, subalgebra, ideal.

AMS *Subject Class.* (2010): 17B05, 17B99.

1. Introduction

In this paper we classify the subspaces of each (real) four-dimensional Lie algebra; two subspaces Γ_1 and Γ_2 of a Lie algebra $\mathfrak{g}$ are equivalent if there exists a Lie algebra automorphism $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot \Gamma_1 = \Gamma_2$. The subspaces are enumerated and partitioned into the subalgebras (which are not ideals), the ideals, the subspaces generating proper subalgebras, and the full-rank subspaces (i.e., those subspaces generating the entire Lie algebra). Furthermore, the quotients by the one-dimensional fully characteristic ideals are determined. The decomposable algebras are covered in Section 2 while the indecomposable algebras are covered in Section 3. The classification procedure (utilizing computer algebra for verification of completeness and nonredundancy) is described in Appendix B; a typical proof is also supplied.

We prefer to use (a modified version of) the enumeration of the four-dimensional Lie algebras due to Mubarakzyanov ([16]), similar to that used by Patera et al. ([18, 17]); details are given in Appendix A. Also, we shall find it convenient to represent these algebras as subalgebras of $\mathfrak{gl}(n, \mathbb{R})$, $n \leq 4$ (matrix representations of low dimensional Lie algebras are given in [11]).

For each Lie algebra, the corresponding enumeration of subspaces is catalogued as follows:

SA:	subalgebras (which are not ideals)
I:	ideals (which are not characteristic)
CI:	characteristic ideals (which are not fully characteristic)
FCI:	fully characteristic ideals
GSA:	subspaces generating proper subalgebras
FRSS:	full-rank subspaces.

(A characteristic ideal is an ideal which is invariant under all derivations whereas a fully characteristic ideal is one which is invariant under all automorphisms.) We refer to this partitioning of the subspaces as the *subspace structure* of the Lie algebra. Unless stated otherwise, each listed subalgebra is Abelian. For example, the oscillator algebra $\mathfrak{g}_{4.9}^0$ has the following subspace structure:

$$\begin{array}{ll} \text{SA:} & \langle E_2 \rangle , \quad \langle E_4 \rangle , \quad \langle E_1, E_2 \rangle , \quad \langle E_1, E_4 \rangle \\ \text{FCI:} & \langle E_1 \rangle , \quad \langle E_1, E_2, E_3 \rangle \cong \mathfrak{g}_{3.1} \\ \text{GSA:} & \langle E_2, E_3 \rangle \\ \text{FRSS:} & \langle E_2, E_4 \rangle , \quad \langle E_1, E_2, E_4 \rangle , \quad \langle E_2, E_3, E_4 \rangle . \end{array}$$

(Here E_1, E_2, E_3, E_4 is a basis for $\mathfrak{g}_{4.9}^0$ and $\langle \cdot \rangle$ denotes the linear span.) This means, for instance, that any subalgebra of $\mathfrak{g}_{4.9}^0$ (which is not an ideal) is equivalent to exactly one of the Abelian subalgebras $\langle E_2 \rangle$, $\langle E_4 \rangle$, $\langle E_1, E_2 \rangle$, and $\langle E_1, E_4 \rangle$.

In Section 4, we briefly explore to what extent a classification of the subspaces of a given Lie algebra $\mathfrak{g}$ can be projected (resp. lifted) to a quotient (resp. extension) of $\mathfrak{g}$. A few remarks conclude the paper.

2. Decomposable algebras

2.1. Algebra $\mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1$ (trivial extension of $\mathfrak{aff}(\mathbb{R})$). The Lie algebra

$$\mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ w & -x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 \ : \ w, x, y, z \in \mathbb{R} \right\}$$

has nonzero commutator relations $[E_1, E_2] = E_1$ and center $\{0\} \oplus 2\mathfrak{g}_1$. The group of automorphisms is given by

$$\mathsf{Aut}\,(\mathfrak{g}_{2.1} \oplus 2\mathfrak{g}) = \left\{ \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a_3 & a_4 & a_5 \\ 0 & a_6 & a_7 & a_8 \end{bmatrix} \,:\, a_1, \ldots, a_8 \in \mathbb{R},\, a_1(a_4a_8 - a_5a_7) \neq 0 \right\}.$$

THEOREM 2.1. *The Lie algebra* $\mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1$ *has the following subspace structure:*

SA:	$\langle E_2 \rangle$, $\langle E_1 + E_4 \rangle$, $\langle E_2, E_4 \rangle$, $\langle E_1 + E_3, E_4 \rangle$, $\langle E_2, E_3, E_4 \rangle$
I:	$\langle E_4 \rangle$, $\langle E_1, E_2 \rangle \cong \mathfrak{g}_{2.1}$, $\langle E_1, E_4 \rangle$, $\langle E_1, E_2, E_4 \rangle \cong \mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$
FCI:	$\langle E_1 \rangle$, $\langle E_3, E_4 \rangle$, $\langle E_1, E_3, E_4 \rangle$
GSA:	$\langle E_1 + E_4, E_2 \rangle$
FRSS:	$\langle E_1 + E_3, E_2, E_4 \rangle$.

$\mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1$ is a fully characteristic extension of the three-dimensional Abelian Lie algebra $3\mathfrak{g}_1$. Indeed,

$$q : \mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1 \to 3\mathfrak{g}_1, \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ w & -x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{bmatrix} \longmapsto \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

is a Lie algebra epimorphism with kernel $\ker q = \langle E_1 \rangle$.

2.2. ALGEBRA $2\mathfrak{g}_{2.1}$. The Lie algebra

$$2\mathfrak{g}_{2.1} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ w & -x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & y & -z \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 \;:\; w, x, y, z \in \mathbb{R} \right\}$$

has nonzero commutator relations $[E_1, E_2] = E_1$, $[E_3, E_4] = E_3$ and trivial center. The group of automorphisms is given by

$$\mathsf{Aut}\,(2\mathfrak{g}_{2.1}) = \left\{ \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_3 & a_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & a_3 & a_4 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \right.$$
$$\left. : a_1, \ldots, a_4 \in \mathbb{R},\, a_1a_3 \neq 0 \right\}.$$

THEOREM 2.2. *The Lie algebra* $2\mathfrak{g}_{2.1}$ *has the following subspace structure:*

$$
\begin{array}{ll}
\textit{SA:} & \langle E_4\rangle\,, \quad \langle E_1+E_3\rangle\,, \quad \langle E_1+E_4\rangle\,, \quad \langle \varepsilon E_2+E_4\rangle \\
& \langle E_1,E_4\rangle\,, \quad \langle E_2,E_4\rangle\,, \quad \langle E_1+E_4,E_3\rangle \cong \mathfrak{g}_{2.1} \\
& \langle E_1,\eta E_2+E_4\rangle \cong \mathfrak{g}_{2.1}, \quad \langle E_1+E_3,E_2+E_4\rangle \cong \mathfrak{g}_{2.1} \\
& \langle E_1,E_2,E_4\rangle \cong \mathfrak{g}_1 \oplus \mathfrak{g}_{2.1} \\
\textit{CI:} & \langle E_1\rangle\,, \quad \langle E_1,E_2\rangle \cong \mathfrak{g}_{2.1}, \quad \langle E_1,E_2,E_3\rangle \cong \mathfrak{g}_1 \oplus \mathfrak{g}_{2.1} \\
& \langle E_1,\mu E_2+E_4,E_3\rangle \cong \mathfrak{g}_{3.4}^{\frac{1+\mu}{1-\mu}} \\
\textit{FCI:} & \langle E_1,E_3\rangle\,, \quad \langle E_1,E_2+E_4,E_3\rangle \cong \mathfrak{g}_{3.3} \\
& \langle E_1,-E_2+E_4,E_3\rangle \cong \mathfrak{g}_{3.4}^{0} \\
\textit{GSA:} & \langle E_1+E_3,E_4\rangle\,, \quad \langle E_1+E_4,E_2\rangle\,, \quad \langle E_1+E_3,\gamma E_2+E_4\rangle \\
\textit{FRSS:} & \langle E_1+E_4,E_2+E_3\rangle\,, \quad \langle E_1,E_2+E_3,E_4\rangle\,, \quad \langle E_1+E_3,E_2,E_4\rangle.
\end{array}
$$

Here $\varepsilon,\gamma,\eta,\mu \neq 0$, $-1 \leq \varepsilon \leq 1$, $-1 \leq \gamma < 1$, $-1 < \mu < 1$ *parametrize families of distinct* (*nonequivalent*) *subspaces.*

$2\mathfrak{g}_{2.1}$ has no fully characteristic one-dimensional ideals.

2.3. ALGEBRA $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$ (TRIVIAL EXTENSION OF THE HEISENBERG ALGEBRA). The Lie algebra

$$
\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1 = \left\{ \begin{bmatrix} 0 & x & w & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 \ : \ w,x,y,z \in \mathbb{R} \right\}
$$

has nonzero commutator relations $[E_2,E_3] = E_1$ and center $\langle E_1, E_4\rangle$. The group of automorphisms is given by

$$
\mathsf{Aut}(\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1) = \left\{ \begin{bmatrix} a_2a_7 - a_6a_3 & a_1 & a_5 & a_9 \\ 0 & a_2 & a_6 & 0 \\ 0 & a_3 & a_7 & 0 \\ 0 & a_4 & a_8 & a_{10} \end{bmatrix} \right.
\left. \ : \ a_1, \ldots, a_{10} \in \mathbb{R},\ (a_2a_7 - a_6a_3)a_{10} \neq 0 \right\}.
$$

THEOREM 2.3. *The Lie algebra* $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$ *has the following subspace structure:*

SA:	$\langle E_2 \rangle$, $\langle E_2, E_4 \rangle$
I:	$\langle E_4 \rangle$, $\langle E_1, E_2 \rangle$, $\langle E_1, E_2, E_3 \rangle \cong \mathfrak{g}_{3.1}$, $\langle E_1, E_2, E_4 \rangle$
FCI:	$\langle E_1 \rangle$, $\langle E_1, E_4 \rangle$
GSA:	$\langle E_2, E_3 \rangle$
FRSS:	$\langle E_2, E_3, E_4 \rangle$.

$\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$ is a fully characteristic (central) extension of the three-dimensional Abelian Lie algebra $3\mathfrak{g}_1$. Indeed,

$$q : \mathfrak{g}_{3.1} \oplus \mathfrak{g}_1 \to 3\mathfrak{g}_1, \qquad \begin{bmatrix} 0 & x & w & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z \end{bmatrix} \longmapsto \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

is a Lie algebra epimorphism with kernel $\ker q = \langle E_1 \rangle$.

2.4. Algebra $\mathfrak{g}_{3.2} \oplus \mathfrak{g}_1$. The Lie algebra

$$\mathfrak{g}_{3.2} \oplus \mathfrak{g}_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ x & y & 0 & 0 \\ w & -y & y & 0 \\ 0 & 0 & 0 & z \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 \ : \ w, x, y, z \in \mathbb{R} \right\}$$

has nonzero commutator relations $[E_2, E_3] = E_1 - E_2$, $[E_3, E_1] = E_1$ and center $\{0\} \oplus \mathfrak{g}_1$. The group of automorphisms is given by

$$\mathsf{Aut}(\mathfrak{g}_{3.2} \oplus \mathfrak{g}_1) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ 0 & a_1 & a_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_5 & a_6 \end{bmatrix} \ : \ a_1, \ldots, a_6 \in \mathbb{R},\ a_1 a_6 \neq 0 \right\}.$$

Theorem 2.4. *The Lie algebra $\mathfrak{g}_{3.2} \oplus \mathfrak{g}_1$ has the following subspace structure:*

SA:	$\langle E_2 \rangle$, $\langle E_3 \rangle$, $\langle E_1 + E_4 \rangle$, $\langle E_2 + E_4 \rangle$
	$\langle E_1, E_3 \rangle \cong \mathfrak{g}_{2.1}$, $\langle E_2, E_4 \rangle$, $\langle E_3, E_4 \rangle$
	$\langle E_1, E_2 + E_4 \rangle$, $\langle E_1 + E_4, E_2 \rangle$, $\langle E_1, E_3, E_4 \rangle \cong \mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$
I:	$\langle E_1, E_2, E_3 \rangle \cong \mathfrak{g}_{3.2}$
FCI:	$\langle E_1 \rangle$, $\langle E_4 \rangle$, $\langle E_1, E_2 \rangle$, $\langle E_1, E_4 \rangle$, $\langle E_1, E_2, E_4 \rangle$
GSA:	$\langle E_2, E_3 \rangle$, $\langle E_1 + E_4, E_3 \rangle$
FRSS:	$\langle E_2 + E_4, E_3 \rangle$
	$\langle E_2, E_3, E_4 \rangle$, $\langle E_1 + E_4, E_2, E_3 \rangle$, $\langle E_1, E_2 + E_4, E_3 \rangle$.

Clearly $\mathfrak{g}_{3.2} \oplus \mathfrak{g}_1$ is a trivial fully characteristic (central) extension of $\mathfrak{g}_{3.2}$. However, it is also a fully characteristic extension of $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$. Indeed,

$$q : \mathfrak{g}_{3.2} \oplus \mathfrak{g}_1 \to \mathfrak{g}_{2.1} \oplus \mathfrak{g}_1, \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ x & y & 0 & 0 \\ w & -y & y & 0 \\ 0 & 0 & 0 & z \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ x & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

is a Lie algebra epimorphism with kernel $\ker q = \langle E_1 \rangle$.

2.5. Algebra $\mathfrak{g}_{3.3} \oplus \mathfrak{g}_1$. The Lie algebra

$$\mathfrak{g}_{3.3} \oplus \mathfrak{g}_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ x & y & 0 & 0 \\ w & 0 & y & 0 \\ 0 & 0 & 0 & z \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 \ : \ w, x, y, z \in \mathbb{R} \right\}$$

has nonzero commutators $[E_2, E_3] = -E_2$, $[E_3, E_1] = E_1$ and center $\{0\} \oplus \mathfrak{g}_1$. The group of automorphisms is given by

$$\mathsf{Aut}(\mathfrak{g}_{3.3} \oplus \mathfrak{g}_1) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ a_4 & a_5 & a_6 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_7 & a_8 \end{bmatrix} \ : \ a_1, \ldots, a_8 \in \mathbb{R},\ (a_1a_5 - a_2a_4)a_8 \neq 0 \right\}.$$

Theorem 2.5. *The Lie algebra $\mathfrak{g}_{3.3} \oplus \mathfrak{g}_1$ has the following subspace structure:*

SA: $\langle E_3 \rangle$, $\langle E_1 + E_4 \rangle$, $\langle E_1, E_3 \rangle \cong \mathfrak{g}_{2.1}$, $\langle E_3, E_4 \rangle$
$\langle E_1, E_2 + E_4 \rangle$, $\langle E_1, E_3, E_4 \rangle \cong \mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$
I: $\langle E_1 \rangle$, $\langle E_1, E_4 \rangle$, $\langle E_1, E_2, E_3 \rangle \cong \mathfrak{g}_{3.3}$
FCI: $\langle E_4 \rangle$, $\langle E_1, E_2 \rangle$, $\langle E_1, E_2, E_4 \rangle$
GSA: $\langle E_1 + E_4, E_3 \rangle$
FRSS: $\langle E_1, E_2 + E_4, E_3 \rangle$.

Clearly $\mathfrak{g}_{3.3} \oplus \mathfrak{g}_1$ is a trivial fully characteristic (central) extension of $\mathfrak{g}_{3.3}$.

2.6. Algebra $\mathfrak{g}_{3.4}^0 \oplus \mathfrak{g}_1$ (trivial extension of the semi-Euclidean algebra). The Lie algebra

$$\mathfrak{g}_{3.4}^0 \oplus \mathfrak{g}_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ w & 0 & -y & 0 \\ x & -y & 0 & 0 \\ 0 & 0 & 0 & z \end{bmatrix} \ : \ w, x, y, z \in \mathbb{R} \right\}$$

has nonzero commutator relations $[E_2, E_3] = E_1,\ [E_3, E_1] = -E_2$ and center $\{0\} \oplus \mathfrak{g}_1$. The group of automorphisms is given by

$$\mathsf{Aut}\,(\mathfrak{g}_{3.3} \oplus \mathfrak{g}_1) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ \sigma\, a_2 & \sigma\, a_1 & a_4 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & a_5 & a_6 \end{bmatrix} \right. \\ \left. : \ a_1, \ldots, a_6 \in \mathbb{R},\, \sigma = \pm 1,\, (a_1^2 - a_2^2)a_6 \neq 0 \right\}.$$

THEOREM 2.6. *The Lie algebra* $\mathfrak{g}_{3.4}^0 \oplus \mathfrak{g}_1$ *has the following subspace structure:*

SA:	$\langle E_1 \rangle\,,\quad \langle E_3 \rangle\,,\quad \langle E_1 + E_4 \rangle\,,\quad \langle E_1 + E_2 + E_4 \rangle$
	$\langle E_1, E_4 \rangle\,,\quad \langle E_3, E_4 \rangle\,,\quad \langle E_1, E_2 + E_4 \rangle\,,\ \langle E_1 + E_2, E_3 \rangle \cong \mathfrak{g}_{2.1}$
	$\langle E_1 + E_2, E_1 + E_4 \rangle\,,\quad \langle E_1 + E_2, E_3, E_4 \rangle \cong \mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$
I:	$\langle E_1, E_2, E_3 \rangle \cong \mathfrak{g}_{3.4}^0$
CI:	$\langle E_1 + E_2 \rangle\,,\quad \langle E_1 + E_2, E_4 \rangle$
FCI:	$\langle E_4 \rangle\,,\quad \langle E_1, E_2 \rangle\,,\quad \langle E_1, E_2, E_4 \rangle$
GSA:	$\langle E_1, E_3 \rangle\,,\quad \langle E_1 + E_2 + E_4, E_3 \rangle$
FRSS:	$\langle E_1 + E_4, E_3 \rangle$
	$\langle E_1, E_3, E_4 \rangle\,,\quad \langle E_1, E_2 + E_4, E_3 \rangle\,,\quad \langle E_1 + E_2, E_1 + E_4, E_3 \rangle.$

Clearly $\mathfrak{g}_{3.4}^0 \oplus \mathfrak{g}_1$ is a trivial fully characteristic (central) extension of $\mathfrak{g}_{3.4}^0$.

2.7. ALGEBRA $\mathfrak{g}_{3.4}^\alpha \oplus \mathfrak{g}_1$. The Lie algebra

$$\mathfrak{g}_{3.4}^\alpha \oplus \mathfrak{g}_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ w & \alpha\, y & -y & 0 \\ x & -y & \alpha\, y & 0 \\ 0 & 0 & 0 & z \end{bmatrix} \ : \ w, x, y, z \in \mathbb{R} \right\}$$

has nonzero commutators $[E_2, E_3] = E_1 - \alpha E_2,\ [E_3, E_1] = \alpha E_1 - E_2$ and center $\{0\} \oplus \mathfrak{g}_1$. Here $\alpha > 0,\ \alpha \neq 1$. (When $\alpha = 0$, we recover $\mathfrak{g}_{3.4}^0 \oplus \mathfrak{g}_1$ and when $\alpha = 1$, we recover $\mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1$.) The group of automorphisms is given by

$$\mathsf{Aut}(\mathfrak{g}_{3.4}^\alpha \oplus \mathfrak{g}_1) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ a_2 & a_1 & a_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_5 & a_6 \end{bmatrix} \ : \ a_1, \ldots, a_6 \in \mathbb{R},\, (a_1^2 - a_2^2)a_6 \neq 0 \right\}.$$

Remark 2.7. $\mathsf{Aut}(\mathfrak{g}_{3.4}^{\alpha} \oplus \mathfrak{g}_1)$ is a subgroup of $\mathsf{Aut}(\mathfrak{g}_{3.4}^{0} \oplus \mathfrak{g}_1)$. Indeed, $\mathsf{Aut}(\mathfrak{g}_{3.4}^{0} \oplus \mathfrak{g}_1)$ decomposes as a semidirect product of subgroups

$$\mathsf{Aut}(\mathfrak{g}_{3.4}^{0} \oplus \mathfrak{g}_1) = \mathsf{Aut}(\mathfrak{g}_{3.4}^{\alpha} \oplus \mathfrak{g}_1) \rtimes \{\mathrm{diag}(1, \sigma, \sigma, 1) \,:\, \sigma = \pm 1\}\,.$$

Accordingly, the classification of the subspaces of $\mathfrak{g}_{3.4}^{\alpha} \oplus \mathfrak{g}_1$ is very similar to that of $\mathfrak{g}_{3.4}^{0} \oplus \mathfrak{g}_1$. Indeed, any subspace of $\mathfrak{g}_{3.4}^{\alpha} \oplus \mathfrak{g}_1$ is equivalent to a subspace with the same formal expression as that of one of $\mathfrak{g}_{3.4}^{0} \oplus \mathfrak{g}_1$, up to a transformation $\mathrm{diag}(1, \sigma, \sigma, 1)$, $\sigma = \pm 1$.

Theorem 2.8. *The Lie algebra* $\mathfrak{g}_{3.4}^{\alpha} \oplus \mathfrak{g}_1$ *has the following subspace structure:*

$$\begin{array}{lll}
\textit{SA:} & & \langle E_1 \rangle\,,\ \langle E_3 \rangle\,,\ \langle E_1 + E_4 \rangle\,,\quad \langle E_1 + E_2 + E_4 \rangle\,,\quad \langle E_1 - E_2 + E_4 \rangle \\
& & \langle E_1, E_4 \rangle\,,\quad \langle E_3, E_4 \rangle\,,\quad \langle E_1, E_2 + E_4 \rangle\,,\quad \langle E_1 + E_2, E_3 \rangle \cong \mathfrak{g}_{2.1} \\
& & \langle E_1 - E_2, E_3 \rangle \cong \mathfrak{g}_{2.1},\ \langle E_1 + E_2, E_1 + E_4 \rangle\,,\ \langle E_1 - E_2, E_1 + E_4 \rangle \\
& & \langle E_1 + E_2, E_3, E_4 \rangle \cong \mathfrak{g}_{2.1} \oplus \mathfrak{g}_1,\quad \langle E_1 - E_2, E_3, E_4 \rangle \cong \mathfrak{g}_{2.1} \oplus \mathfrak{g}_1 \\
\textit{I:} & & \langle E_1, E_2, E_3 \rangle \cong \mathfrak{g}_{3.4}^{\alpha} \\
& \textit{FCI:} & \langle E_4 \rangle\,,\quad \langle E_1 + E_2 \rangle\,,\quad \langle E_1 - E_2 \rangle \\
& & \langle E_1, E_2 \rangle\,,\quad \langle E_1 + E_2, E_4 \rangle\,,\quad \langle E_1 - E_2, E_4 \rangle\,,\quad \langle E_1, E_2, E_4 \rangle \\
\textit{GSA:} & & \langle E_1, E_3 \rangle\,,\quad \langle E_1 + E_2 + E_4, E_3 \rangle\,,\quad \langle E_1 - E_2 + E_4, E_3 \rangle \\
\textit{FRSS:} & & \langle E_1 + E_4, E_3 \rangle\,,\quad \langle E_1, E_3, E_4 \rangle\,,\quad \langle E_1, E_2 + E_4, E_3 \rangle \\
& & \langle E_1 + E_2, E_1 + E_4, E_3 \rangle\,,\quad \langle E_1 - E_2, E_1 + E_4, E_3 \rangle.
\end{array}$$

Clearly $\mathfrak{g}_{3.4}^{\alpha} \oplus \mathfrak{g}_1$ is a trivial fully characteristic (central) extension of $\mathfrak{g}_{3.4}^{\alpha}$. However, it is also a fully characteristic extension of $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$. Indeed, the mappings

$$q_1 : \mathfrak{g}_{3.4}^{\alpha} \oplus \mathfrak{g}_1 \to \mathfrak{g}_{2.1} \oplus \mathfrak{g}_1, \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ w & \alpha y & -y & 0 \\ x & -y & \alpha y & 0 \\ 0 & 0 & 0 & z \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ w - x & (\alpha + 1) y & 0 \\ 0 & 0 & z \end{bmatrix}$$

$$q_2 : \mathfrak{g}_{3.4}^{\alpha} \oplus \mathfrak{g}_1 \to \mathfrak{g}_{2.1} \oplus \mathfrak{g}_1, \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ w & \alpha y & -y & 0 \\ x & -y & \alpha y & 0 \\ 0 & 0 & 0 & z \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ w + x & (\alpha - 1) y & 0 \\ 0 & 0 & z \end{bmatrix}$$

are Lie algebra epimorphisms with kernels $\ker q_1 = \langle E_1 + E_2 \rangle$ and $\ker q_2 = \langle E_1 - E_2 \rangle$, respectively.

2.8. ALGEBRA $\mathfrak{g}_{3.5}^0 \oplus \mathfrak{g}_1$ (TRIVIAL EXTENSION OF THE EUCLIDEAN ALGEBRA). The Lie algebra

$$\mathfrak{g}_{3.5}^0 \oplus \mathfrak{g}_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ w & 0 & -y & 0 \\ x & y & 0 & 0 \\ 0 & 0 & 0 & z \end{bmatrix} \ : \ w, x, y, z \in \mathbb{R} \right\}$$

has nonzero commutator relations $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$ and center $\{0\} \oplus \mathfrak{g}_1$. The group of automorphisms is given by

$$\mathsf{Aut}\,(\mathfrak{g}_{3.5}^0 \oplus \mathfrak{g}_1) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ -\sigma a_2 & \sigma a_1 & a_4 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & a_5 & a_6 \end{bmatrix} \right. \\ \left. : \ a_1, \ldots, a_6 \in \mathbb{R},\ (a_1^2 + a_2^2) a_6 \neq 0,\ \sigma = \pm 1 \right\}.$$

THEOREM 2.9. *The Lie algebra $\mathfrak{g}_{3.5}^0 \oplus \mathfrak{g}_1$ has the following subspace structure:*

SA:	$\langle E_1 \rangle$, $\langle E_3 \rangle$, $\langle E_1 + E_4 \rangle$, $\langle E_1, E_4 \rangle$, $\langle E_3, E_4 \rangle$, $\langle E_1, E_2 + E_4 \rangle$
I:	$\langle E_1, E_2, E_3 \rangle \cong \mathfrak{g}_{3.5}^0$
FCI:	$\langle E_4 \rangle$, $\langle E_1, E_2 \rangle$, $\langle E_1, E_2, E_4 \rangle$
GSA:	$\langle E_1, E_3 \rangle$
FRSS:	$\langle E_1 + E_4, E_3 \rangle$, $\langle E_1, E_3, E_4 \rangle$, $\langle E_1, E_2 + E_4, E_3 \rangle$.

Clearly $\mathfrak{g}_{3.5}^0 \oplus \mathfrak{g}_1$ is a trivial fully characteristic (central) extension of $\mathfrak{g}_{3.5}^0$.

2.9. ALGEBRA $\mathfrak{g}_{3.5}^\alpha \oplus \mathfrak{g}_1$. The Lie algebra

$$\mathfrak{g}_{3.5}^\alpha \oplus \mathfrak{g}_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ w & \alpha\, y & -y & 0 \\ x & y & \alpha\, y & 0 \\ 0 & 0 & 0 & z \end{bmatrix} \ : \ w, x, y, z \in \mathbb{R} \right\}$$

has nonzero commutator relations $[E_2, E_3] = E_1 - \alpha E_2$, $[E_3, E_1] = \alpha E_1 + E_2$ and center $\{0\} \oplus \mathfrak{g}_1$. Here $\alpha > 0$. The group of automorphisms is given by

$$\mathsf{Aut}\,(\mathfrak{g}_{3.5}^\alpha \oplus \mathfrak{g}_1) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ -a_2 & a_1 & a_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_5 & a_6 \end{bmatrix} \ : \ a_1, \ldots, a_6 \in \mathbb{R}, (a_1^2 + a_2^2) a_6 \neq 0 \right\}.$$

Remark 2.10. $\mathsf{Aut}(\mathfrak{g}_{3.5}^{\alpha} \oplus \mathfrak{g}_1)$ is a subgroup of $\mathsf{Aut}(\mathfrak{g}_{3.5}^{0} \oplus \mathfrak{g}_1)$. Indeed, $\mathsf{Aut}(\mathfrak{g}_{3.5}^{0} \oplus \mathfrak{g}_1)$ decomposes as a semidirect product of subgroups

$$\mathsf{Aut}(\mathfrak{g}_{3.5}^{0} \oplus \mathfrak{g}_1) = \mathsf{Aut}(\mathfrak{g}_{3.5}^{\alpha} \oplus \mathfrak{g}_1) \rtimes \{\mathrm{diag}(1, \sigma, \sigma, 1) \, : \, \sigma = \pm 1\} \, .$$

Accordingly, the classification of the subspaces of $\mathfrak{g}_{3.5}^{\alpha} \oplus \mathfrak{g}_1$ is very similar to that of $\mathfrak{g}_{3.5}^{0} \oplus \mathfrak{g}_1$. In fact, the classification of subspaces turns out to be formally identical.

THEOREM 2.11. *The Lie algebra $\mathfrak{g}_{3.5}^{\alpha} \oplus \mathfrak{g}_1$ has the following subspace structure:*

SA:	$\langle E_1 \rangle$, $\langle E_3 \rangle$, $\langle E_1 + E_4 \rangle$, $\langle E_1, E_4 \rangle$, $\langle E_3, E_4 \rangle$, $\langle E_1, E_2 + E_4 \rangle$
I:	$\langle E_1, E_2, E_3 \rangle \cong \mathfrak{g}_{3.5}^{\alpha}$
FCI:	$\langle E_4 \rangle$, $\langle E_1, E_2 \rangle$, $\langle E_1, E_2, E_4 \rangle$
GSA:	$\langle E_1, E_3 \rangle$
FRSS:	$\langle E_1 + E_4, E_3 \rangle$, $\langle E_1, E_3, E_4 \rangle$, $\langle E_1, E_2 + E_4, E_3 \rangle$.

Clearly $\mathfrak{g}_{3.5}^{\alpha} \oplus \mathfrak{g}_1$ is a trivial fully characteristic (central) extension of $\mathfrak{g}_{3.5}^{\alpha}$.

2.10. ALGEBRA $\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$ (TRIVIAL EXTENSION OF THE PSEUDO-ORTHOGONAL ALGEBRA). The Lie algebra

$$\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1 = \left\{ \begin{bmatrix} \frac{z+w}{2} & \frac{x-y}{2} \\ \frac{x+y}{2} & \frac{z-w}{2} \end{bmatrix} \, : \, w, x, y, z \in \mathbb{R} \right\} = \mathfrak{gl}(2, \mathbb{R})$$

has nonzero commutator relations $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, $[E_1, E_2] = -E_3$, and center $\{0\} \oplus \mathfrak{g}_1$. The group of automorphisms is given by

$$\mathsf{Aut}(\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1) = \left\{ \begin{bmatrix} & & & 0 \\ & g & & 0 \\ & & & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix} \, : \, g \in \mathsf{SO}\,(2,1),\ a_4 \in \mathbb{R},\ a_4 \neq 0 \right\}$$

where

$$\mathsf{SO}\,(2,1) = \left\{ g \in \mathbb{R}^{3 \times 3} \, : \, g^\top J g = J,\ \det g = 1 \right\}, \qquad J = \mathrm{diag}(1, 1, -1).$$

THEOREM 2.12. *The Lie algebra $\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$ has the following subspace structure:*

SA: $\langle E_1\rangle$, $\langle E_3\rangle$, $\langle E_1+E_3\rangle$, $\langle E_1+E_4\rangle$, $\langle E_3+E_4\rangle$
$\langle E_1+E_3+E_4\rangle$, $\langle E_1,E_4\rangle$, $\langle E_3,E_4\rangle$, $\langle E_1,E_2+E_3\rangle \cong \mathfrak{g}_{2.1}$
$\langle E_2+E_3,E_4\rangle$, $\langle E_1+E_4,E_2+E_3\rangle \cong \mathfrak{g}_{2.1}$
$\langle E_1+E_3,E_2,E_4\rangle \cong \mathfrak{g}_{2.1}\oplus\mathfrak{g}_1$
FCI: $\langle E_4\rangle$, $\langle E_1,E_2,E_3\rangle \cong \mathfrak{g}_{3.6}$
GSA: $\langle E_1,E_2\rangle$, $\langle E_1,E_3\rangle$, $\langle E_1,E_2+E_3+E_4\rangle$
FRSS: $\langle E_1,E_2+E_4\rangle$, $\langle E_1,E_3+E_4\rangle$, $\langle E_1+E_4,E_3\rangle$
$\langle E_2+E_4,E_2+E_3\rangle$, $\langle E_1,E_2,E_4\rangle$, $\langle E_2,E_3,E_4\rangle$
$\langle E_1,E_2,E_3+E_4\rangle$, $\langle E_1+E_4,E_2,E_3\rangle$, $\langle E_1+E_3,E_2,E_1+E_4\rangle$.

Clearly $\mathfrak{g}_{3.6}\oplus\mathfrak{g}_1$ is a trivial fully characteristic (central) extension of $\mathfrak{g}_{3.6}$.

2.11. ALGEBRA $\mathfrak{g}_{3.7}\oplus\mathfrak{g}_1$ (TRIVIAL EXTENSION OF THE ORTHOGONAL ALGEBRA). The Lie algebra

$$\mathfrak{g}_{3.7}\oplus\mathfrak{g}_1 = \left\{\begin{bmatrix} 0 & w & -x & 0\\ -w & 0 & y & 0\\ x & -y & 0 & 0\\ 0 & 0 & 0 & z\end{bmatrix} \,:\, w,x,y,z\in\mathbb{R}\right\}$$

has nonzero commutator relations $[E_2,E_3]=E_1$, $[E_3,E_1]=E_2$, $[E_1,E_2]=E_3$ and center $\{0\}\oplus\mathfrak{g}_1$. The group of automorphisms is given by

$$\mathsf{Aut}\,(\mathfrak{g}_{3.7}\oplus\mathfrak{g}_1) = \left\{\begin{bmatrix} & & & 0\\ & g & & 0\\ & & & 0\\ 0 & 0 & 0 & a_4\end{bmatrix} \,:\, g\in\mathsf{SO}\,(3),\ a_4\in\mathbb{R},\ a_4\neq 0\right\}$$

where $\mathsf{SO}\,(3) = \left\{g\in\mathbb{R}^{3\times 3} \,:\, g^\top g = I_3,\ \det g = 1\right\}$.

THEOREM 2.13. *The Lie algebra $\mathfrak{g}_{3.7}\oplus\mathfrak{g}_1$ has the following subspace structure:*

SA: $\langle E_1\rangle$, $\langle E_1+E_4\rangle$, $\langle E_1,E_4\rangle$
FCI: $\langle E_4\rangle$, $\langle E_1,E_2,E_3\rangle \cong \mathfrak{g}_{3.7}$
GSA: $\langle E_1,E_2\rangle$
FRSS: $\langle E_1,E_2+E_4\rangle$, $\langle E_2,E_3,E_4\rangle$, $\langle E_1+E_4,E_2,E_3\rangle$.

Clearly $\mathfrak{g}_{3.7}\oplus\mathfrak{g}_1$ is a trivial fully characteristic (central) extension of $\mathfrak{g}_{3.7}$.

3. Indecomposable algebras

3.1. Algebra $\mathfrak{g}_{4.1}$ (Engel algebra, central extension of the Heisenberg algebra). The Lie algebra

$$\mathfrak{g}_{4.1} = \left\{ \begin{bmatrix} 0 & z & 0 & w \\ 0 & 0 & z & z-x \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 \ : \ w,x,y,z \in \mathbb{R} \right\}$$

has nonzero commutators $[E_2, E_4] = E_1$, $[E_3, E_4] = E_2$ and center $\langle E_1 \rangle$. The group of automorphisms is given by

$$\mathsf{Aut}\,(\mathfrak{g}_{4.1}) = \left\{ \begin{bmatrix} a_1 a_2^2 & a_2 a_3 & a_4 & a_5 \\ 0 & a_1 a_2 & a_3 & a_6 \\ 0 & 0 & a_1 & a_7 \\ 0 & 0 & 0 & a_2 \end{bmatrix} \ : \ a_1, \ldots, a_7 \in \mathbb{R},\ a_1 a_2 \neq 0 \right\}.$$

Theorem 3.1. *The Lie algebra* $\mathfrak{g}_{4.1}$ *has the following subspace structure:*

SA: $\langle E_2 \rangle$, $\langle E_3 \rangle$, $\langle E_4 \rangle$, $\langle E_1, E_3 \rangle$, $\langle E_1, E_4 \rangle$, $\langle E_2, E_3 \rangle$
I: $\langle E_1, E_2, E_4 \rangle \cong \mathfrak{g}_{3.1}$
FCI: $\langle E_1 \rangle$, $\langle E_1, E_2 \rangle$, $\langle E_1, E_2, E_3 \rangle$
GSA: $\langle E_2, E_4 \rangle$
FRSS: $\langle E_3, E_4 \rangle$, $\langle E_1, E_3, E_4 \rangle$, $\langle E_2, E_3, E_4 \rangle$.

$\mathfrak{g}_{4.1}$ is a fully characteristic (central) extension of the Heisenberg algebra $\mathfrak{g}_{3.1}$. Indeed, the mapping

$$q : \mathfrak{g}_{4.1} \to \mathfrak{g}_{3.1}, \qquad \begin{bmatrix} 0 & z & 0 & w \\ 0 & 0 & z & z-x \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & y & x \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_1 \rangle = \mathrm{Z}(\mathfrak{g}_{4.1})$.

3.2. Algebra $\mathfrak{g}_{4.2}^{\alpha}$. The Lie algebra

$$\mathfrak{g}_{4.2}^{\alpha} = \left\{ \begin{bmatrix} -\alpha z & 0 & 0 & w \\ 0 & -z & -\alpha z & \alpha x \\ 0 & 0 & -z & y \\ 0 & 0 & 0 & 0 \end{bmatrix} \right.$$

$$\left. = wE_1 + xE_2 + yE_3 + zE_4 : w,x,y,z \in \mathbb{R} \right\}$$

has nonzero commutators $[E_1, E_4] = \alpha E_1,\ [E_2, E_4] = E_2,\ [E_3, E_4] = E_2 + E_3$ and trivial center. Here $\alpha \neq 0$. (We note that $\alpha = 0$ corresponds to the Lie algebra $\mathfrak{g}_{3.2} \oplus \mathfrak{g}_1$.) If $\alpha \neq 1$, then the group of automorphisms is given by

$$\mathsf{Aut}\,(\mathfrak{g}_{4.2}^{\alpha}) = \left\{ \begin{bmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_2 & a_3 & a_5 \\ 0 & 0 & a_2 & a_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} : a_1, \ldots, a_6 \in \mathbb{R} \right\}.$$

If $\alpha = 1$, then we have

$$\mathsf{Aut}\,(\mathfrak{g}_{4.2}^{\alpha}) = \left\{ \begin{bmatrix} a_1 & 0 & a_4 & a_6 \\ a_2 & a_3 & a_5 & a_7 \\ 0 & 0 & a_3 & a_8 \\ 0 & 0 & 0 & 1 \end{bmatrix} : a_1, \ldots, a_8 \in \mathbb{R} \right\}.$$

3.2.1. Case $\alpha \neq 1$

Theorem 3.2. *The Lie algebra $\mathfrak{g}_{4.2}^{\alpha}$, $\alpha \neq 1$ has the following subspace structure:*

SA:	$\langle E_3 \rangle, \quad \langle E_4 \rangle, \quad \langle E_1 + E_2 \rangle, \quad \langle E_1 + E_3 \rangle, \quad \langle E_1, E_3 \rangle$
	$\langle E_1, E_4 \rangle \cong \mathfrak{g}_{2.1}, \quad \langle E_2, E_4 \rangle \cong \mathfrak{g}_{2.1}, \quad \langle E_1 + E_2, E_3 \rangle, \quad \langle E_1 + E_3, E_2 \rangle$
	$\langle E_1, E_2, E_4 \rangle \cong \mathfrak{g}_{3.4}^{\beta}, \quad \langle E_2, E_3, E_4 \rangle \cong \mathfrak{g}_{3.2}$
FCI:	$\langle E_1 \rangle, \quad \langle E_2 \rangle, \quad \langle E_1, E_2 \rangle, \quad \langle E_2, E_3 \rangle, \quad \langle E_1, E_2, E_3 \rangle$
GSA:	$\langle E_3, E_4 \rangle, \quad \langle E_1 + E_2, E_4 \rangle$
FRSS:	$\langle E_1 + E_3, E_4 \rangle, \langle E_1, E_3, E_4 \rangle, \langle E_1 + E_2, E_3, E_4 \rangle, \langle E_1 + E_3, E_2, E_4 \rangle.$

Here $\beta = \frac{1+\alpha}{1-\alpha}$ when $-1 \leq \alpha < 1$ and $\beta = \frac{\alpha+1}{\alpha-1}$ when $|\alpha| > 1$.

$\mathfrak{g}_{4.2}^{\alpha}$, $\alpha \neq 1$ is a fully characteristic extension of the Lie algebra $\mathfrak{g}_{3.2}$. Indeed, the mapping

$$q : \mathfrak{g}_{4.2}^{\alpha} \to \mathfrak{g}_{3.2}, \qquad \begin{bmatrix} -\alpha z & 0 & 0 & w \\ 0 & -z & -\alpha z & \alpha x \\ 0 & 0 & -z & y \\ 0 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ -y & -z & 0 \\ x & z & -z \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_1 \rangle$. If $-1 \leq \alpha < 1$, then $\mathfrak{g}_{4.2}^{\alpha}$ is a fully-characteristic extension of $\mathfrak{g}_{3.4}^{\beta}$ where $\beta = \frac{1+\alpha}{1-\alpha}$. Indeed, the mapping $q : \mathfrak{g}_{4.2}^{\alpha} \to \mathfrak{g}_{3.4}^{\frac{1+\alpha}{1-\alpha}}$,

$$\begin{bmatrix} -\alpha z & 0 & 0 & w \\ 0 & -z & -\alpha z & \alpha x \\ 0 & 0 & -z & y \\ 0 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ w+y & -\frac{1}{2}z(\alpha+1) & -\frac{1}{2}z(\alpha-1) \\ w-y & -\frac{1}{2}z(\alpha-1) & -\frac{1}{2}z(\alpha+1) \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_2 \rangle$. If $|\alpha| > 1$, then $\mathfrak{g}_{4.2}^{\alpha}$ is a fully-characteristic extension of $\mathfrak{g}_{3.4}^{\beta}$ where $\beta = \frac{\alpha+1}{\alpha-1}$. Indeed, the mapping $q : \mathfrak{g}_{4.2}^{\alpha} \to \mathfrak{g}_{3.4}^{\frac{\alpha+1}{\alpha-1}}$,

$$\begin{bmatrix} -\alpha z & 0 & 0 & w \\ 0 & -z & -\alpha z & \alpha x \\ 0 & 0 & -z & y \\ 0 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ w+y & -\frac{1}{2}z(\alpha+1) & \frac{1}{2}z(\alpha-1) \\ -w+y & \frac{1}{2}z(\alpha-1) & -\frac{1}{2}z(\alpha+1) \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_2 \rangle$.

3.2.2. Case $\alpha = 1$

Theorem 3.3. *The Lie algebra $\mathfrak{g}_{4.2}^{1}$ has the following subspace structure:*

SA:	$\langle E_3 \rangle$, $\langle E_4 \rangle$, $\langle E_1, E_3 \rangle$, $\langle E_1, E_4 \rangle \cong \mathfrak{g}_{2.1}$, $\langle E_2, E_4 \rangle \cong \mathfrak{g}_{2.1}$
	$\langle E_1, E_2, E_4 \rangle \cong \mathfrak{g}_{3.3}$, $\langle E_2, E_3, E_4 \rangle \cong \mathfrak{g}_{3.2}$
I:	$\langle E_1 \rangle$, $\langle E_2, E_3 \rangle$
FCI:	$\langle E_2 \rangle$, $\langle E_1, E_2 \rangle$, $\langle E_1, E_2, E_3 \rangle$
GSA:	$\langle E_3, E_4 \rangle$
FRSS:	$\langle E_1, E_3, E_4 \rangle$.

$\mathfrak{g}_{4.2}^{1}$ is a fully characteristic extension of the Lie algebra $\mathfrak{g}_{3.3}$. Indeed, the mapping

$$q : \mathfrak{g}_{4.2}^{1} \to \mathfrak{g}_{3.3}, \qquad \begin{bmatrix} -z & 0 & 0 & w \\ 0 & -z & -z & x \\ 0 & 0 & -z & y \\ 0 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ y & -z & 0 \\ w & 0 & -z \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_2 \rangle$.

3.3. Algebra $\mathfrak{g}_{4.3}$. The Lie algebra

$$\mathfrak{g}_{4.3} = \left\{ \begin{bmatrix} -z & 0 & 0 & w \\ 0 & 0 & -z & x \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 \ : \ w, x, y, z \in \mathbb{R} \right\}$$

has nonzero commutator relations $[E_1, E_4] = E_1$, $[E_3, E_4] = E_2$ and center $\langle E_2 \rangle$. The group of automorphisms is given by

$$\mathsf{Aut}(\mathfrak{g}_{4.3}) = \left\{ \begin{bmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_2 & a_3 & a_5 \\ 0 & 0 & a_2 & a_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ : \ a_1, \ldots, a_6 \in \mathbb{R}, \ a_1 a_2 \neq 0 \right\}.$$

Theorem 3.4. *The Lie algebra $\mathfrak{g}_{4.3}$ has the following subspace structure:*

SA: $\langle E_3 \rangle$, $\langle E_4 \rangle$, $\langle E_1 + E_2 \rangle$, $\langle E_1 + E_3 \rangle$
$\langle E_1, E_3 \rangle$, $\langle E_1, E_4 \rangle \cong \mathfrak{g}_{2.1}$, $\langle E_2, E_4 \rangle$,
$\langle E_1 + E_2, E_3 \rangle$, $\langle E_1 + E_3, E_2 \rangle$, $\langle E_2, E_3, E_4 \rangle \cong \mathfrak{g}_{3.1}$
I: $\langle E_1, E_2, E_4 \rangle \cong \mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$
FCI: $\langle E_1 \rangle$, $\langle E_2 \rangle$, $\langle E_1, E_2 \rangle$, $\langle E_2, E_3 \rangle$, $\langle E_1, E_2, E_3 \rangle$
GSA: $\langle E_3, E_4 \rangle$, $\langle E_1 + E_2, E_4 \rangle$
FRSS: $\langle E_1 + E_3, E_4 \rangle$, $\langle E_1, E_3, E_4 \rangle$
$\langle E_1 + E_2, E_3, E_4 \rangle$, $\langle E_1 + E_3, E_2, E_4 \rangle$.

$\mathfrak{g}_{4.3}$ is a fully characteristic extension of the Lie algebra $\mathfrak{g}_{3.1}$. Indeed, the mapping

$$q : \mathfrak{g}_{4.3} \to \mathfrak{g}_{3.1}, \qquad \begin{bmatrix} -z & 0 & 0 & w \\ 0 & 0 & -z & x \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & y & x \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_1 \rangle$. The Lie algebra $\mathfrak{g}_{4.3}$ is also a fully characteristic (central) extension of the Lie algebra $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$. Indeed, the mapping

$$q : \mathfrak{g}_{4.3} \to \mathfrak{g}_{2.1} \oplus \mathfrak{g}_1, \qquad \begin{bmatrix} -z & 0 & 0 & w \\ 0 & 0 & -z & x \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ w & -z & 0 \\ 0 & 0 & y \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_2 \rangle = \mathrm{Z}(\mathfrak{g}_{4.3})$.

3.4. Algebra $\mathfrak{g}_{4.4}$. The Lie algebra

$$\mathfrak{g}_{4.4} = \left\{ \begin{bmatrix} -z & -z & 0 & w \\ 0 & -z & -z & x \\ 0 & 0 & -z & y \\ 0 & 0 & 0 & 0 \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 \, : \, w, x, y, z \in \mathbb{R} \right\}$$

has nonzero commutator relations $[E_1, E_4] = E_1$, $[E_2, E_4] = E_1 + E_2$, $[E_3, E_4] = E_2 + E_3$ and trivial center. The group of automorphisms is given by

$$\mathsf{Aut}(\mathfrak{g}_{4.4}) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_1 & a_2 & a_5 \\ 0 & 0 & a_1 & a_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \, : \, a_1, \ldots, a_6 \in \mathbb{R},\, a_1 \neq 0 \right\}.$$

Theorem 3.5. *The Lie algebra $\mathfrak{g}_{4.4}$ has the following subspace structure:*

SA: $\langle E_2 \rangle$, $\langle E_3 \rangle$, $\langle E_4 \rangle$
$\langle E_1, E_3 \rangle$, $\langle E_1, E_4 \rangle \cong \mathfrak{g}_{2.1}$, $\langle E_2, E_3 \rangle$, $\langle E_1, E_2, E_4 \rangle \cong \mathfrak{g}_{3.2}$
FCI: $\langle E_1 \rangle$, $\langle E_1, E_2 \rangle$, $\langle E_1, E_2, E_3 \rangle$
GSA: $\langle E_2, E_4 \rangle$
FRSS: $\langle E_3, E_4 \rangle$, $\langle E_1, E_3, E_4 \rangle$, $\langle E_2, E_3, E_4 \rangle$.

$\mathfrak{g}_{4.4}$ is a fully characteristic extension of the Lie algebra $\mathfrak{g}_{3.2}$. Indeed, the mapping

$$q : \mathfrak{g}_{4.4} \to \mathfrak{g}_{3.2}, \qquad \begin{bmatrix} -z & -z & 0 & w \\ 0 & -z & -z & x \\ 0 & 0 & -z & y \\ 0 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ y & -z & 0 \\ -x & z & -z \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_1 \rangle$.

3.5. Algebra $\mathfrak{g}_{4.5}^{\alpha,\beta}$. The Lie algebra

$$\mathfrak{g}_{4.5}^{\alpha,\beta} = \left\{ \begin{bmatrix} -z & 0 & 0 & w \\ 0 & -\alpha\, z & 0 & y \\ 0 & 0 & -\beta\, z & x \\ 0 & 0 & 0 & 0 \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 : \, w, x, y, z \in \mathbb{R} \right\}$$

has nonzero commutator relations $[E_1, E_4] = E_1,\ [E_2, E_4] = \beta E_2,\ [E_3, E_4] = \alpha E_3$ and trivial center. Here $-1 < \alpha \leq \beta \leq 1,\ \alpha\beta \neq 0$ or $\alpha = -1,\ 0 < \beta \leq 1$. We note that

$$\mathfrak{g}_{4.5}^{\beta,\alpha} \cong \mathfrak{g}_{4.5}^{\alpha,\beta}, \qquad \mathfrak{g}_{4.5}^{\alpha,\beta} \cong \mathfrak{g}_{4.5}^{\frac{\alpha}{\beta},\frac{1}{\beta}}, \qquad \mathfrak{g}_{4.5}^{\alpha,\beta} \cong \mathfrak{g}_{4.5}^{\frac{1}{\alpha},\frac{\beta}{\alpha}}, \qquad \mathfrak{g}_{4.5}^{-1,\beta} \cong \mathfrak{g}_{4.5}^{-1,-\beta},$$
$$\mathfrak{g}_{4.5}^{\frac{\gamma-1}{\gamma+1},0} \cong \mathfrak{g}_{3.4}^{\gamma} \oplus \mathfrak{g}_1, \qquad \mathfrak{g}_{4.5}^{0,0} \cong \mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1, \qquad \mathfrak{g}_{4,5}^{1,0} \cong \mathfrak{g}_{3.3} \oplus \mathfrak{g}_1.$$

If $\alpha \neq 1,\ \beta \neq 1$ and $\alpha \neq \beta$, then the group of automorphisms is given by

$$\mathsf{Aut}\,(\mathfrak{g}_{4.5}^{\alpha,\beta}) = \left\{ \begin{bmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_2 & 0 & a_5 \\ 0 & 0 & a_3 & a_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ : \ a_1, \ldots, a_6 \in \mathbb{R},\ a_1 a_2 a_3 \neq 0 \right\}.$$

If $\alpha \neq 1$ and $\alpha = \beta$, then the group of automorphisms is given by

$$\mathsf{Aut}\,(\mathfrak{g}_{4.5}^{\alpha,\beta}) = \left\{ \begin{bmatrix} a_1 & 0 & 0 & a_6 \\ 0 & a_2 & a_4 & a_7 \\ 0 & a_3 & a_5 & a_8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ : \ a_1, \ldots, a_8 \in \mathbb{R},\ a_1(a_2 a_5 - a_3 a_4) \neq 0 \right\}.$$

If $\alpha \neq 1$ and $\beta = 1$, then the group of automorphisms is given by

$$\mathsf{Aut}\,(\mathfrak{g}_{4.5}^{\alpha,\beta}) = \left\{ \begin{bmatrix} a_1 & a_3 & 0 & a_6 \\ a_2 & a_4 & 0 & a_7 \\ 0 & 0 & a_5 & a_8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ : \ a_1, \ldots, a_8 \in \mathbb{R}, (a_1 a_4 - a_2 a_3) a_5 \neq 0 \right\}.$$

If $\alpha = 1$ (and $\beta = 1$), then the group of automorphisms is given by

$$\mathsf{Aut}\,(\mathfrak{g}_{4.5}^{\alpha,\beta}) = \left\{ \begin{bmatrix} a_1 & a_4 & a_7 & a_{10} \\ a_2 & a_5 & a_8 & a_{11} \\ a_3 & a_6 & a_9 & a_{12} \\ 0 & 0 & 0 & 1 \end{bmatrix} \ : \ a_1, \ldots, a_{12} \in \mathbb{R}, \begin{vmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{vmatrix} \neq 0 \right\}.$$

3.5.1. Case $\alpha \neq 1,\ \beta \neq 1,\ \alpha \neq \beta$

Theorem 3.6. *The Lie algebra* $\mathfrak{g}_{4.5}^{\alpha,\beta},\ \alpha \neq 1,\ \beta \neq 1,\ \alpha \neq \beta$ *has the following subspace structure:*

$$\begin{array}{rl}
\textit{SA:} & \langle E_4\rangle\,, \quad \langle E_1+E_2\rangle\,, \quad \langle E_1+E_3\rangle\,, \quad \langle E_2+E_3\rangle\,, \quad \langle E_1+E_2+E_3\rangle \\
& \langle E_1,E_4\rangle \cong \mathfrak{g}_{2.1}, \quad \langle E_2,E_4\rangle \cong \mathfrak{g}_{2.1}, \quad \langle E_3,E_4\rangle \cong \mathfrak{g}_{2.1}, \quad \langle E_1+E_2,E_3\rangle \\
& \langle E_1+E_3,E_2\rangle\,, \quad \langle E_1,E_2+E_3\rangle\,, \quad \langle E_1-E_2,E_1+E_2+E_3\rangle \\
& \langle E_1,E_2,E_4\rangle \cong \mathfrak{g}_{3.4}^{\frac{1+\beta}{1-\beta}}, \quad \langle E_1,E_3,E_4\rangle \cong \mathfrak{g}_{3.4}^{\frac{1+\alpha}{1-\alpha}}, \quad \langle E_2,E_3,E_4\rangle \cong \mathfrak{g}_{3.4}^{\chi} \\
\textit{FCI:} & \langle E_1\rangle\,, \quad \langle E_2\rangle\,, \quad \langle E_3\rangle \\
& \langle E_1,E_2\rangle\,, \quad \langle E_1,E_3\rangle\,, \quad \langle E_2,E_3\rangle\,, \quad \langle E_1,E_2,E_3\rangle \\
\textit{GSA:} & \langle E_1+E_2,E_4\rangle\,, \quad \langle E_1+E_3,E_4\rangle\,, \quad \langle E_2+E_3,E_4\rangle \\
\textit{FRSS:} & \langle E_1+E_2+E_3,E_4\rangle\,, \quad \langle E_1,E_2+E_3,E_4\rangle\,, \quad \langle E_1+E_3,E_2,E_4\rangle \\
& \langle E_1+E_2,E_3,E_4\rangle\,, \quad \langle E_1-E_2,E_1+E_2+E_3,E_4\rangle.
\end{array}$$

Here $\chi = \frac{\alpha+\beta}{\beta-\alpha}$ *if* $\alpha+\beta \geq 0$ *and* $\chi = -\frac{\alpha+\beta}{\beta-\alpha}$ *if* $\alpha+\beta < 0$.

If $\alpha+\beta \geq 0$, then $\mathfrak{g}_{4.5}^{\alpha,\beta}$ is a fully characteristic extension of the Lie algebra $\mathfrak{g}_{3.4}^{\chi}$ with $\chi = \frac{\alpha+\beta}{\beta-\alpha}$. Indeed, the mapping $q : \mathfrak{g}_{4.5}^{\alpha,\beta} \to \mathfrak{g}_{3.4}^{\frac{\alpha+\beta}{\beta-\alpha}}$,

$$\begin{bmatrix} -z & 0 & 0 & w \\ 0 & -z\alpha & 0 & y \\ 0 & 0 & -z\beta & x \\ 0 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ y+\frac{x}{\beta} & -\frac{1}{2}z(\alpha+\beta) & \frac{1}{2}z(-\alpha+\beta) \\ y-\frac{x}{\beta} & \frac{1}{2}z(-\alpha+\beta) & -\frac{1}{2}z(\alpha+\beta) \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_1\rangle$. If $\alpha+\beta < 0$, then $\mathfrak{g}_{4.5}^{\alpha,\beta}$ is a fully characteristic extension of the Lie algebra $\mathfrak{g}_{3.4}^{\chi}$ with $\chi = \frac{\alpha+\beta}{\alpha-\beta}$. Indeed, the mapping $q : \mathfrak{g}_{4.5}^{\alpha,\beta} \to \mathfrak{g}_{3.4}^{\frac{\alpha+\beta}{\alpha-\beta}}$,

$$\begin{bmatrix} -z & 0 & 0 & w \\ 0 & -z\alpha & 0 & y \\ 0 & 0 & -z\beta & x \\ 0 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ y+\frac{x}{\beta} & -\frac{1}{2}z(\alpha+\beta) & \frac{1}{2}z(\alpha-\beta) \\ -y+\frac{x}{\beta} & \frac{1}{2}z(\alpha-\beta) & -\frac{1}{2}z(\alpha+\beta) \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_1\rangle$. The Lie algebra $\mathfrak{g}_{4.5}^{\alpha,\beta}$ is a fully characteristic extension of the Lie algebra $\mathfrak{g}_{3.4}^{\frac{1+\alpha}{1-\alpha}}$. Indeed, the mapping $q : \mathfrak{g}_{4.5}^{\alpha,\beta} \to \mathfrak{g}_{3.4}^{\frac{1+\alpha}{1-\alpha}}$,

$$\begin{bmatrix} -z & 0 & 0 & w \\ 0 & -z\alpha & 0 & y \\ 0 & 0 & -z\beta & x \\ 0 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ w+y & -\frac{1}{2}z(1+\alpha) & -\frac{1}{2}z(-1+\alpha) \\ -w+y & -\frac{1}{2}z(-1+\alpha) & -\frac{1}{2}z(1+\alpha) \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_2\rangle$. Furthermore, $\mathfrak{g}_{4.5}^{\alpha,\beta}$ is also a fully characteristic extension of $\mathfrak{g}_{3.4}^{\frac{1+\beta}{1-\beta}}$. Indeed, the mapping $q : \mathfrak{g}_{4.5}^{\alpha,\beta} \to \mathfrak{g}_{3.4}^{\frac{1+\beta}{1-\beta}}$,

$$\begin{bmatrix} -z & 0 & 0 & w \\ 0 & -z\alpha & 0 & y \\ 0 & 0 & -z\beta & x \\ 0 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ w+x & -\frac{1}{2}z(1+\beta) & -\frac{1}{2}z(-1+\beta) \\ -w+x & -\frac{1}{2}z(-1+\beta) & -\frac{1}{2}z(1+\beta) \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_3\rangle$.

3.5.2. Case $\alpha \neq 1,\ \alpha = \beta$

Theorem 3.7. *The Lie algebra $\mathfrak{g}_{4.5}^{\alpha,\beta}$, $\alpha \neq 1$, $\alpha = \beta$ has the following subspace structure:*

SA:	$\langle E_4\rangle$, $\langle E_1+E_2\rangle$, $\langle E_1, E_4\rangle \cong \mathfrak{g}_{2.1}$, $\langle E_2, E_4\rangle \cong \mathfrak{g}_{2.1}$
	$\langle E_1+E_2, E_3\rangle$, $\langle E_1, E_2, E_4\rangle \cong \mathfrak{g}_{3.4}^{\frac{1+\alpha}{1-\alpha}}$, $\langle E_2, E_3, E_4\rangle \cong \mathfrak{g}_{3.3}$
I:	$\langle E_2\rangle$, $\langle E_1, E_2\rangle$
FCI:	$\langle E_1\rangle$, $\langle E_2, E_3\rangle$, $\langle E_1, E_2, E_3\rangle$
GSA:	$\langle E_1+E_2, E_4\rangle$
FRSS:	$\langle E_1+E_2, E_3, E_4\rangle$.

$\mathfrak{g}_{4.5}^{\alpha,\beta}$, $\alpha \neq 1$, $\alpha = \beta$ is a fully characteristic extension of the Lie algebra $\mathfrak{g}_{3.3}$. Indeed, the mapping

$$q : \mathfrak{g}_{4.5}^{\alpha,\alpha} \to \mathfrak{g}_{3.3}, \qquad \begin{bmatrix} -z & 0 & 0 & w \\ 0 & -z\alpha & 0 & y \\ 0 & 0 & -z\alpha & x \\ 0 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ \frac{x+y}{\alpha} & -z\alpha & 0 \\ \frac{x-y}{\alpha} & 0 & -z\alpha \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_1\rangle$.

3.5.3. Case $\alpha \neq 1,\ \beta = 1$

Theorem 3.8. *The Lie algebra $\mathfrak{g}_{4.5}^{\alpha,1}$, $\alpha \neq 1$ has the following subspace structure:*

$$\begin{array}{ll}
\textit{SA:} & \langle E_4\rangle\,, \quad \langle E_1+E_3\rangle\,, \quad \langle E_1,E_4\rangle \cong \mathfrak{g}_{2.1}, \quad \langle E_3,E_4\rangle \cong \mathfrak{g}_{2.1}, \\
 & \langle E_1,E_2+E_3\rangle\,, \quad \langle E_1,E_2,E_4\rangle \cong \mathfrak{g}_{3.3}, \quad \langle E_1,E_3,E_4\rangle \cong \mathfrak{g}_{3.4}^{\frac{1+\alpha}{1-\alpha}} \\
\textit{I:} & \langle E_1\rangle\,, \quad \langle E_1,E_3\rangle \\
\textit{FCI:} & \langle E_3\rangle\,, \quad \langle E_1,E_2\rangle\,, \quad \langle E_1,E_2,E_3\rangle \\
\textit{GSA:} & \langle E_1+E_3,E_4\rangle \\
\textit{FRSS:} & \langle E_1,E_2+E_3,E_4\rangle.
\end{array}$$

$\mathfrak{g}_{4.5}^{\alpha,1}$, $\alpha \neq 1$ is a fully characteristic extension of the Lie algebra $\mathfrak{g}_{3.4}^{\frac{1+\alpha}{1-\alpha}}$. Indeed, the mapping $q : \mathfrak{g}_{4.5}^{\alpha,1} \to \mathfrak{g}_{3.4}^{\frac{1+\alpha}{1-\alpha}}$,

$$\begin{bmatrix} -z & 0 & 0 & w \\ 0 & -\alpha z & 0 & y \\ 0 & 0 & -z & x \\ 0 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ x+y & -\frac{1}{2}z(1+\alpha) & -\frac{1}{2}z(-1+\alpha) \\ -x+y & -\frac{1}{2}z(-1+\alpha) & -\frac{1}{2}z(1+\alpha) \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_1\rangle$.

3.5.4. CASE $\alpha = 1$

THEOREM 3.9. *The Lie algebra* $\mathfrak{g}_{4.5}^{1,1}$ *has the following subspace structure:*

$$\begin{array}{ll}
\textit{SA:} & \langle E_4\rangle\,, \quad \langle E_1,E_4\rangle \cong \mathfrak{g}_{2.1}, \quad \langle E_1,E_2,E_4\rangle \cong \mathfrak{g}_{3.3} \\
\textit{I:} & \langle E_1\rangle\,, \quad \langle E_1,E_2\rangle \\
\textit{FCI:} & \langle E_1,E_2,E_3\rangle.
\end{array}$$

Every subspace of $\mathfrak{g}_{4.5}^{1,1}$ is a subalgebra; hence $\mathfrak{g}_{4.5}^{1,1}$ admits no proper full-rank subspaces. Also, $\mathfrak{g}_{4.5}^{1,1}$ has no one-dimensional fully characteristic ideals. Hence $\mathfrak{g}_{4.5}^{1,1}$ is not a fully characteristic extension of any three-dimensional Lie algebra.

3.6. ALGEBRA $\mathfrak{g}_{4.6}^{\alpha,\beta}$. The Lie algebra

$$\mathfrak{g}_{4.6}^{\alpha,\beta} = \left\{ \begin{bmatrix} -\alpha z & 0 & 0 & w \\ 0 & -\beta z & -z & -y \\ 0 & z & -\beta z & x \\ 0 & 0 & 0 & 0 \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 \;:\; w,x,y,z \in \mathbb{R} \right\}$$

has nonzero commutator relations $[E_1, E_4] = \alpha E_1$, $[E_2, E_4] = \beta E_2 - E_3$, $[E_3, E_4] = E_2 + \beta E_3$ and trivial center. Here $\alpha > 0$ and $\beta \in \mathbb{R}$. (We note that $\mathfrak{g}_{4.6}^{0,\beta} \cong \mathfrak{g}_{3.5}^{|\beta|} \oplus \mathfrak{g}_1$ and $\mathfrak{g}_{4.6}^{\alpha,\beta} \cong \mathfrak{g}_{4.6}^{-\alpha,-\beta}$.) The group of automorphisms is given by

$$\mathsf{Aut}\,(\mathfrak{g}_{4.6}^{\alpha,\beta}) = \left\{ \begin{bmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_2 & a_3 & a_5 \\ 0 & -a_3 & a_2 & a_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \;:\; a_1, \ldots, a_6 \in \mathbb{R},\, a_1(a_2^2 + a_3^2) \neq 0 \right\}.$$

THEOREM 3.10. *The Lie algebra $\mathfrak{g}_{4.6}^{\alpha,\beta}$ has the following subspace structure:*

SA:	$\langle E_2 \rangle$, $\langle E_4 \rangle$, $\langle E_1 + E_2 \rangle$, $\langle E_1, E_2 \rangle$, $\langle E_1, E_4 \rangle \cong \mathfrak{g}_{2.1}$
	$\langle E_1 + E_2, E_3 \rangle$, $\langle E_2, E_3, E_4 \rangle \cong \mathfrak{g}_{3.5}^{\lvert\beta\rvert}$
FCI:	$\langle E_1 \rangle$, $\langle E_2, E_3 \rangle$, $\langle E_1, E_2, E_3 \rangle$
GSA:	$\langle E_2, E_4 \rangle$
FRSS:	$\langle E_1 + E_2, E_4 \rangle$, $\langle E_1, E_2, E_4 \rangle$, $\langle E_1 + E_2, E_3, E_4 \rangle$.

$\mathfrak{g}_{4.6}^{\alpha,\beta}$ is a fully characteristic extension of the Lie algebra $\mathfrak{g}_{3.5}^{|\beta|}$. Indeed, the mappings

$$q_1 : \mathfrak{g}_{4.6}^{\alpha,\beta} \to \mathfrak{g}_{3.5}^{\beta}, \qquad \begin{bmatrix} -z\alpha & 0 & 0 & w \\ 0 & -z\beta & -z & -y \\ 0 & z & -z\beta & x \\ 0 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ y & -z\beta & z \\ x & -z & -z\beta \end{bmatrix} \qquad \beta \geq 0$$

$$q_2 : \mathfrak{g}_{4.6}^{\alpha,\beta} \to \mathfrak{g}_{3.5}^{-\beta}, \qquad \begin{bmatrix} -z\alpha & 0 & 0 & w \\ 0 & -z\beta & -z & -y \\ 0 & z & -z\beta & x \\ 0 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ x & -z\beta & -z \\ y & z & -z\beta \end{bmatrix} \qquad \beta < 0$$

are epimorphisms with $\ker q_1 = \langle E_1 \rangle$ and $\ker q_2 = \langle E_1 \rangle$.

3.7. ALGEBRA $\mathfrak{g}_{4.7}$.

The Lie algebra

$$\mathfrak{g}_{4.7} = \left\{ \begin{bmatrix} -2z & -y & x & 2w \\ 0 & -z & -z & x \\ 0 & 0 & -z & y \\ 0 & 0 & 0 & 0 \end{bmatrix} \;:\; w, x, y, z \in \mathbb{R} \right\}$$

has nonzero commutators $[E_2, E_3] = E_1$, $[E_1, E_4] = 2E_1$, $[E_2, E_4] = E_2$, $[E_3, E_4] = E_2 + E_3$ and trivial center. The group of automorphisms is given by

$$\mathsf{Aut}(\mathfrak{g}_{4.7}) = \left\{ \begin{bmatrix} a_1^2 & -a_1a_3 & a_1a_4 - (a_1 + a_2)\, a_3 & a_5 \\ 0 & a_1 & a_2 & a_4 \\ 0 & 0 & a_1 & a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right. \\ \left. : a_1, \ldots, a_5 \in \mathbb{R},\ a_1 \neq 0 \right\}.$$

THEOREM 3.11. *The Lie algebra $\mathfrak{g}_{4.7}$ has the following subspace structure:*

$$\begin{array}{ll} \textit{SA:} & \langle E_2 \rangle, \quad \langle E_3 \rangle, \quad \langle E_4 \rangle, \quad \langle E_1, E_3 \rangle \\ & \langle E_1, E_4 \rangle \cong \mathfrak{g}_{2.1}, \quad \langle E_2, E_4 \rangle \cong \mathfrak{g}_{2.1}, \quad \langle E_1, E_2, E_4 \rangle \cong \mathfrak{g}_{3.4}^{3} \\ \textit{FCI:} & \langle E_1 \rangle, \quad \langle E_1, E_2 \rangle, \quad \langle E_1, E_2, E_3 \rangle \cong \mathfrak{g}_{3.1} \\ \textit{GSA:} & \langle E_2, E_3 \rangle, \quad \langle E_1 + E_2, E_4 \rangle \\ \textit{FRSS:} & \langle E_3, E_4 \rangle, \quad \langle E_1, E_3, E_4 \rangle, \quad \langle E_2, E_3, E_4 \rangle. \end{array}$$

$\mathfrak{g}_{4.7}$ is a fully characteristic extension of the Lie algebra $\mathfrak{g}_{3.2}$. Indeed, the mapping

$$q : \mathfrak{g}_{4.7} \to \mathfrak{g}_{3.2}, \qquad \begin{bmatrix} -2z & -y & x & 2w \\ 0 & -z & -z & x \\ 0 & 0 & -z & y \\ 0 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ -y & -z & 0 \\ x & z & -z \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_1 \rangle$.

3.8. ALGEBRA $\mathfrak{g}_{4.8}^{-1}$ (CENTRAL EXTENSION OF THE SEMI-EUCLIDEAN ALGEBRA). The Lie algebra

$$\mathfrak{g}_{4.8}^{-1} = \left\{ \begin{bmatrix} 0 & x & w \\ 0 & z & y \\ 0 & 0 & 0 \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 \ : \ w, x, y, z \in \mathbb{R} \right\}$$

has nonzero commutator relations $[E_2, E_3] = E_1$, $[E_2, E_4] = E_2$, $[E_3, E_4] = -E_3$ and center $\langle E_1 \rangle$. The group of automorphisms is given by

$$\mathsf{Aut}(\mathfrak{g}_{4.8}^{-1}) = \left\{ \begin{bmatrix} a_1a_2 & a_1a_3 & a_2a_4 & a_5 \\ 0 & a_1 & 0 & a_4 \\ 0 & 0 & a_2 & a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -a_1a_2 & -a_2a_4 & -a_1a_3 & a_5 \\ 0 & 0 & a_1 & a_4 \\ 0 & a_2 & 0 & a_3 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right.$$
$$\left. : a_1, \ldots, a_5 \in \mathbb{R}, \quad a_1a_2 \neq 0 \right\}.$$

THEOREM 3.12. (CF. [2]) *The Lie algebra* $\mathfrak{g}_{4.8}^{-1}$ *has the following subspace structure:*

SA: $\langle E_2 \rangle$, $\langle E_2 + E_3 \rangle$, $\langle E_4 \rangle$, $\langle E_1, E_4 \rangle$
$\langle E_2, E_4 \rangle \cong \mathfrak{g}_{2.1}$, $\langle E_1, E_2 + E_3 \rangle$, $\langle E_1, E_2, E_4 \rangle \cong \mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$
CI: $\langle E_1, E_2 \rangle$
FCI: $\langle E_1 \rangle$, $\langle E_1, E_2, E_3 \rangle \cong \mathfrak{g}_{3.1}$
GSA: $\langle E_2, E_3 \rangle$, $\langle E_1 + E_2, E_4 \rangle$
FRSS: $\langle E_2 + E_3, E_4 \rangle$, $\langle E_2, E_3, E_4 \rangle$, $\langle E_1, E_2 + E_3, E_4 \rangle$.

$\mathfrak{g}_{4.8}^{-1}$ is a fully characteristic (central) extension of the semi-Euclidean algebra $\mathfrak{se}(1,1) = \mathfrak{g}_{3.4}^{0}$. Indeed, the mapping

$$q : \mathfrak{g}_{4.8}^{-1} \to \mathfrak{g}_{3.4}^{0}, \qquad \begin{bmatrix} 0 & x & w \\ 0 & z & y \\ 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ x+y & 0 & -z \\ x-y & -z & 0 \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_1 \rangle = \mathrm{Z}(\mathfrak{g}_{4.8}^{-1})$.

3.9. ALGEBRA $\mathfrak{g}_{4.8}^{\alpha}$. The Lie algebra

$$\mathfrak{g}_{4.8}^{\alpha} = \left\{ \begin{bmatrix} -(1+\alpha)\, z & x & w \\ 0 & -\alpha\, z & y \\ 0 & 0 & 0 \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 : w, x, y, z \in \mathbb{R} \right\}$$

has nonzero commutator relations $[E_2, E_3] = E_1$, $[E_1, E_4] = (1+\alpha)E_1$, $[E_2, E_4] = E_2$, $[E_3, E_4] = \alpha E_3$ and trivial center. Here $-1 < \alpha \leq 1$. If $\alpha \neq 0$ and $\alpha \neq 1$, then the group of automorphisms is given by

$$\mathsf{Aut}(\mathfrak{g}_{4.8}^{\alpha}) = \left\{ \begin{bmatrix} a_1a_2 & -a_1a_3 & a_2a_4 & a_5 \\ 0 & a_1 & 0 & a_4 \\ 0 & 0 & a_2 & \alpha\, a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} : a_1, \ldots, a_5 \in \mathbb{R}, a_1a_2 \neq 0 \right\}.$$

If $\alpha = 0$ or $\alpha = 1$, then

$$\mathsf{Aut}(\mathfrak{g}_{4.8}^{0}) = \left\{ \begin{bmatrix} a_1a_2 & a_3 & a_2a_4 & a_5 \\ 0 & a_1 & 0 & a_4 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; : \; a_1, \ldots, a_5 \in \mathbb{R}, \; a_1a_2 \neq 0 \right\}$$

$$\mathsf{Aut}(\mathfrak{g}_{4.8}^{1}) = \left\{ \begin{bmatrix} a_1a_2 - a_6a_7 & -a_1a_3 + a_4a_6 & a_2a_4 - a_3a_7 & a_5 \\ 0 & a_1 & a_7 & a_4 \\ 0 & a_6 & a_2 & a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right. \\ \left. : \; a_1, \ldots, a_7 \in \mathbb{R}, \quad a_1a_2 - a_6a_7 \neq 0 \right\}$$

respectively.

3.9.1. Case $\alpha \neq 0, \; \alpha \neq 1$

Theorem 3.13. *The Lie algebra* $\mathfrak{g}_{4.8}^{\alpha}, \; \alpha \neq 0, \; \alpha \neq 1$ *has the following subspace structure:*

SA:	$\langle E_2 \rangle, \quad \langle E_3 \rangle, \quad \langle E_4 \rangle, \quad \langle E_2 + E_3 \rangle$
	$\langle E_1, E_4 \rangle \cong \mathfrak{g}_{2.1}, \quad \langle E_2, E_4 \rangle \cong \mathfrak{g}_{2.1}, \quad \langle E_3, E_4 \rangle \cong \mathfrak{g}_{2.1}, \quad \langle E_1, E_2 + E_3 \rangle$
	$\langle E_1, E_2, E_4 \rangle \cong \mathfrak{g}_{3.4}^{\left\lvert 1 + \frac{2}{\alpha} \right\rvert}, \quad \langle E_1, E_3, E_4 \rangle \cong \mathfrak{g}_{3.4}^{\lvert 1 + 2\alpha \rvert}$
FCI:	$\langle E_1 \rangle, \quad \langle E_1, E_2 \rangle, \quad \langle E_1, E_3 \rangle, \quad \langle E_1, E_2, E_3 \rangle \cong \mathfrak{g}_{3.1}$
GSA:	$\langle E_2, E_3 \rangle, \quad \langle E_1 + E_2, E_4 \rangle, \quad \langle E_1 + E_3, E_4 \rangle$
FRSS:	$\langle E_2 + E_3, E_4 \rangle, \quad \langle E_2, E_3, E_4 \rangle, \quad \langle E_1, E_2 + E_3, E_4 \rangle.$

$\mathfrak{g}_{4.8}^{\alpha}, \; \alpha \neq 0, \; \alpha \neq 1$ is a fully characteristic extension of the Lie algebra $\mathfrak{g}_{3.4}^{\frac{1+\alpha}{1-\alpha}}$. Indeed, the mapping $q : \mathfrak{g}_{4.8}^{\alpha} \to \mathfrak{g}_{3.4}^{\frac{1+\alpha}{1-\alpha}}$,

$$\begin{bmatrix} -z(1+\alpha) & x & w \\ 0 & -z\alpha & y \\ 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ x + y & -\frac{1}{2}z(1+\alpha) & \frac{1}{2}z(1-\alpha) \\ -x + y & \frac{1}{2}z(1-\alpha) & -\frac{1}{2}z(1+\alpha) \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_1 \rangle$.

3.9.2. Case $\alpha = 0$

Theorem 3.14. *The Lie algebra* $\mathfrak{g}_{4.8}^0$ *has the following subspace structure:*

$$\begin{array}{ll}
\textit{SA:} & \langle E_2\rangle, \quad \langle E_3\rangle, \quad \langle E_4\rangle, \quad \langle E_2+E_3\rangle, \quad \langle E_3+E_4\rangle, \quad \langle E_1,E_4\rangle \cong \mathfrak{g}_{2.1} \\
 & \langle E_2,E_4\rangle \cong \mathfrak{g}_{2.1}, \quad \langle E_3,E_4\rangle, \quad \langle E_1,E_2+E_3\rangle, \quad \langle E_1,E_3+E_4\rangle \cong \mathfrak{g}_{2.1} \\
 & \langle E_1,E_3,E_4\rangle \cong \mathfrak{g}_{2.1}\oplus\mathfrak{g}_1 \\
\textit{I:} & \langle E_1,E_2,E_3+E_4\rangle \cong \mathfrak{g}_{3.2} \\
\textit{FCI:} & \langle E_1\rangle, \quad \langle E_1,E_2\rangle, \quad \langle E_1,E_3\rangle \\
 & \langle E_1,E_2,E_3\rangle \cong \mathfrak{g}_{3.1}, \quad \langle E_1,E_2,E_4\rangle \cong \mathfrak{g}_{3.3} \\
\textit{GSA:} & \langle E_2,E_3\rangle, \quad \langle E_1+E_3,E_4\rangle, \quad \langle E_2,E_3+E_4\rangle \\
\textit{FRSS:} & \langle E_2+E_3,E_4\rangle, \quad \langle E_1,E_2+E_3,E_4\rangle, \quad \langle E_2,E_3,E_4\rangle.
\end{array}$$

$\mathfrak{g}_{4.8}^0$ is a fully characteristic extension of the Lie algebra $\mathfrak{g}_{2.1}\oplus\mathfrak{g}_1$. Indeed, the mapping

$$q : \mathfrak{g}_{4.8}^0 \to \mathfrak{g}_{2.1}\oplus\mathfrak{g}_1, \qquad \begin{bmatrix} -z & x & w \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ x & -z & 0 \\ 0 & 0 & y \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_1\rangle$.

3.9.3. Case $\alpha = 1$

Theorem 3.15. *The Lie algebra* $\mathfrak{g}_{4.8}^1$ *has the following subspace structure:*

$$\begin{array}{ll}
\textit{SA:} & \langle E_2\rangle, \quad \langle E_4\rangle, \quad \langle E_1,E_4\rangle \cong \mathfrak{g}_{2.1}, \quad \langle E_2,E_4\rangle \cong \mathfrak{g}_{2.1}, \quad \langle E_1,E_2,E_4\rangle \cong \mathfrak{g}_{3.4}^3 \\
\textit{I:} & \langle E_1,E_2\rangle \\
\textit{FCI:} & \langle E_1\rangle, \quad \langle E_1,E_2,E_3\rangle \cong \mathfrak{g}_{3.1} \\
\textit{GSA:} & \langle E_1+E_2,E_4\rangle, \quad \langle E_2,E_3\rangle \\
\textit{FRSS:} & \langle E_2,E_3,E_4\rangle.
\end{array}$$

$\mathfrak{g}_{4.8}^1$ is a fully characteristic extension of the Lie algebra $\mathfrak{g}_{3.3}$. Indeed, the mapping

$$q : \mathfrak{g}_{4.8}^1 \to \mathfrak{g}_{3.3}, \qquad \begin{bmatrix} -2z & x & w \\ 0 & -z & y \\ 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ y & -z & 0 \\ x & 0 & -z \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_1\rangle$.

3.10. Algebra $\mathfrak{g}_{4.9}^{0}$ (oscillator algebra, central extension of the Euclidean algebra). The (oscillator) Lie algebra

$$\mathfrak{g}_{4.9}^{0} = \left\{ \begin{bmatrix} 0 & -x & y & -2w \\ 0 & 0 & z & y \\ 0 & -z & 0 & x \\ 0 & 0 & 0 & 0 \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 \ : \ w, x, y, z \in \mathbb{R} \right\}$$

has nonzero commutator relations $[E_2, E_3] = E_1$, $[E_2, E_4] = -E_3$, $[E_3, E_4] = E_2$ and center $\langle E_1 \rangle$. The group of automorphisms is given by

$$\mathsf{Aut}(\mathfrak{g}_{4.9}^{0}) = \left\{ \begin{bmatrix} \sigma\left(a_1^2 + a_2^2\right) & -\sigma a_1 a_4 + a_2 a_5 & -a_1 a_5 - \sigma a_2 a_4 & a_3 \\ 0 & a_1 & a_2 & a_4 \\ 0 & -\sigma a_2 & \sigma a_1 & a_5 \\ 0 & 0 & 0 & \sigma \end{bmatrix} \right.$$
$$\left. : a_1, \ldots, a_5 \in \mathbb{R}, a_1^2 + a_2^2 \neq 0, \ \sigma = \pm 1 \right\}.$$

Theorem 3.16. (cf. [9]) *The Lie algebra $\mathfrak{g}_{4.9}^{0}$ has the following subspace structure:*

SA:	$\langle E_2 \rangle$, $\langle E_4 \rangle$, $\langle E_1, E_2 \rangle$, $\langle E_1, E_4 \rangle$
FCI:	$\langle E_1 \rangle$, $\langle E_1, E_2, E_3 \rangle \cong \mathfrak{g}_{3.1}$
GSA:	$\langle E_2, E_3 \rangle$
FRSS:	$\langle E_2, E_4 \rangle$, $\langle E_1, E_2, E_4 \rangle$, $\langle E_2, E_3, E_4 \rangle$.

$\mathfrak{g}_{4.9}^{0}$ is a fully characteristic (central) extension of the Euclidean algebra $\mathfrak{se}\,(2) = \mathfrak{g}_{3.5}^{0}$. Indeed, the mapping

$$q : \mathfrak{g}_{4.9}^{0} \to \mathfrak{g}_{3.5}^{0}, \qquad \begin{bmatrix} 0 & -x & y & -2w \\ 0 & 0 & z & y \\ 0 & -z & 0 & x \\ 0 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & -z \\ y & z & 0 \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_1 \rangle = \mathrm{Z}(\mathfrak{g}_{4.9}^{0})$.

3.11. Algebra $\mathfrak{g}_{4.9}^{\alpha}$. The Lie algebra

$$\mathfrak{g}_{4.9}^{\alpha} = \left\{ \begin{bmatrix} -2\alpha\, z & -x & y & -2w \\ 0 & -\alpha\, z & z & y \\ 0 & -z & -\alpha\, z & x \\ 0 & 0 & 0 & 0 \end{bmatrix} \right.$$
$$\left. = wE_1 + xE_2 + yE_3 + zE_4 \ : \ w, x, y, z \in \mathbb{R} \right\}$$

has nonzero commutator relations $[E_2, E_3] = E_1$, $[E_1, E_4] = 2\alpha E_1$, $[E_2, E_4] = \alpha E_2 - E_3$, $[E_3, E_4] = E_2 + \alpha E_3$ and trivial center. Here $\alpha > 0$. The group of automorphisms is given by

$$\mathsf{Aut}(\mathfrak{g}_{4.9}^{\alpha}) = \left\{ \begin{bmatrix} a_1^2 + a_2^2 & \frac{*_1}{1+\alpha^2} & \frac{*_2}{1+\alpha^2} & a_3 \\ 0 & a_1 & a_2 & a_4 \\ 0 & -a_2 & a_1 & a_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} : \begin{array}{l} *_1 = -a_2(\alpha\, a_4 - a_5) - a_1(a_4 + \alpha\, a_5) \\ *_2 = a_1(\alpha\, a_4 - a_5) - a_2(a_4 + \alpha\, a_5) \\ a_1, \ldots, a_5 \in \mathbb{R}, \quad a_1^2 + a_2^2 \neq 0 \end{array} \right\}.$$

THEOREM 3.17. *The Lie algebra* $\mathfrak{g}_{4.9}^{\alpha}$ *has the following subspace structure:*

SA: $\langle E_2 \rangle$, $\langle E_4 \rangle$, $\langle E_1, E_2 \rangle$, $\langle E_1, E_4 \rangle \cong \mathfrak{g}_{2.1}$
FCI: $\langle E_1 \rangle$, $\langle E_1, E_2, E_3 \rangle \cong \mathfrak{g}_{3.1}$
GSA: $\langle E_2, E_3 \rangle$
FRSS: $\langle E_2, E_4 \rangle$, $\langle E_1, E_2, E_4 \rangle$, $\langle E_2, E_3, E_4 \rangle$.

$\mathfrak{g}_{4.9}^{\alpha}$ is a fully characteristic extension of the Lie algebra $\mathfrak{g}_{3.5}^{\alpha}$. Indeed, the mapping

$$q : \mathfrak{g}_{4.9}^{\alpha} \to \mathfrak{g}_{3.5}^{\alpha}, \qquad \begin{bmatrix} -2\alpha z & -x & y & -2w \\ 0 & -\alpha z & z & y \\ 0 & -z & -\alpha z & x \\ 0 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ y & -\alpha z & z \\ x & -z & -\alpha z \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_1 \rangle$.

3.12. ALGEBRA $\mathfrak{g}_{4.10}$.

The Lie algebra

$$\mathfrak{g}_{4.10} = \left\{ \begin{bmatrix} -y & z & x \\ -z & -y & w \\ 0 & 0 & 0 \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 \ : \ w, x, y, z \in \mathbb{R} \right\}$$

has nonzero commutator relations $[E_1, E_3] = E_1$, $[E_2, E_3] = E_2$, $[E_1, E_4] = -E_2$, $[E_2, E_4] = E_1$ and trivial center. The group of automorphisms is given by

$$\mathsf{Aut}(\mathfrak{g}_{4.10}) = \left\{ \begin{bmatrix} a_1 & \sigma\, a_2 & a_3 & \sigma\, a_4 \\ -a_2 & \sigma\, a_1 & a_4 & -\sigma\, a_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sigma \end{bmatrix} \;:\; a_1, \ldots, a_4 \in \mathbb{R},\ \sigma = \pm 1,\ a_1 \neq 0 \right\}.$$

Theorem 3.18. *The Lie algebra* $\mathfrak{g}_{4.10}$ *has the following subspace structure:*

SA:	$\langle E_1 \rangle$, $\langle E_3 \rangle$, $\langle \gamma E_3 + E_4 \rangle$, $\langle E_1, E_3 \rangle \cong \mathfrak{g}_{2.1}$, $\langle E_3, E_4 \rangle$
CI:	$\langle E_1, E_2, \eta E_3 + E_4 \rangle \cong \mathfrak{g}_{3.5}^{\eta}$
FCI:	$\langle E_1, E_2 \rangle$, $\langle E_1, E_2, E_3 \rangle \cong \mathfrak{g}_{3.3}$, $\langle E_1, E_2, E_4 \rangle \cong \mathfrak{g}_{3.5}^{0}$
GSA:	$\langle E_1, \gamma E_3 + E_4 \rangle$
FRSS:	$\langle E_1 + E_4, E_3 \rangle$, $\langle E_1, E_3, E_4 \rangle$.

Here $\gamma \geq 0$ *and* $\eta > 0$ *parametrize families of equivalence representatives, each different value yielding a distinct* (*nonequivalent*) *representative.*

$\mathfrak{g}_{4.10}$ has no one-dimensional ideals.

4. Quotients, extensions and equivalence

We briefly explore the relation between the subspaces of a Lie algebra $\mathfrak{g}$ and the subspaces of an extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$. It turns out that if $\widehat{\mathfrak{g}}$ is a fully characteristic extension of $\mathfrak{g}$, then a classification of subspaces of $\mathfrak{g}$ can easily be obtained from a classification of subspaces of $\widehat{\mathfrak{g}}$. Conversely, in some cases a partial classification of subspaces of $\widehat{\mathfrak{g}}$ may be obtained from classification of subspaces of $\mathfrak{g}$. Throughout, let $q : \widehat{\mathfrak{g}} \to \mathfrak{g}$ be an epimorphism (i.e., $\widehat{\mathfrak{g}}$ is an extension of $\mathfrak{g}$ by $\ker q$).

Lemma 4.1. *If* Γ *is a subspace (resp. subalgebra, ideal, full-rank subspace) of* $\widehat{\mathfrak{g}}$, *then* $q(\Gamma)$ *is a subspace (resp. subalgebra, ideal, full-rank subspace) of* $\mathfrak{g}$. *Likewise, if* Γ *is a subspace (resp. subalgebra, ideal, full-rank subspace) of* $\mathfrak{g}$, *then* $q^{-1}(\Gamma)$ *is a subspace (resp. subalgebra, ideal, full-rank subspace) of* $\widehat{\mathfrak{g}}$.

Proof. We prove only the assertion that if Γ is a full-rank subspace of $\mathfrak{g}$, then $q^{-1}(\Gamma)$ is a full-rank subspace of $\widehat{\mathfrak{g}}$ (proofs for the other assertions are rather straightforward). Suppose Γ is a full-rank subspace of $\mathfrak{g}$ and suppose

$q^{-1}(\Gamma)$ is not a full-rank subspace of $\widehat{\mathfrak{g}}$. Then there exists a proper subalgebra $\mathfrak{h}$ of $\widehat{\mathfrak{g}}$ such that $\ker q \subseteq q^{-1}(\Gamma) \subseteq \mathfrak{h} \subset \widehat{\mathfrak{g}}$. Hence $q(q^{-1}(\Gamma)) \subseteq q(\mathfrak{h})$ and so $\Gamma \subseteq q(\mathfrak{h})$. Therefore $\mathsf{Lie}(\Gamma) = \mathfrak{g} \subseteq q(\mathfrak{h})$, i.e., $q(\mathfrak{h}) = \mathfrak{g}$. Let $A \in \widehat{\mathfrak{g}} \backslash \mathfrak{h}$. There exists $B \in \mathfrak{h}$ such that $q(B) = q(A)$. Hence $q(A - B) = 0$ and so $A - B \in \ker q \subseteq \mathfrak{h}$. As $A - B$ and B are both in $\mathfrak{h}$ we have that $A \in \mathfrak{h}$, thus yielding a contradiction. ∎

We can therefore lift and project subspaces by means of the epimorphism q. Next we investigate the compatibility of the automorphisms with the quotient map q.

PROPOSITION 4.2. *Let* $\widehat{\psi} \in \mathsf{Aut}(\widehat{\mathfrak{g}})$. *There exists* $\psi \in \mathsf{Aut}(\mathfrak{g})$ *such that* $q \circ \widehat{\psi} = \psi \circ q$ *if and only if* $\widehat{\psi}(\ker q) = \ker q$.

Proof. Suppose $\widehat{\psi}(\ker q) = \ker q$. As q is surjective, there exists a linear map $p : \mathfrak{g} \to \widehat{\mathfrak{g}}$ such that $q \circ p = \mathrm{id}_{\mathfrak{g}}$. We claim that $\psi = q \circ \widehat{\psi} \circ p$ is an automorphism of $\mathfrak{g}$ satisfying the requirements. Let $A \in \widehat{\mathfrak{g}}$. We have that $q \cdot \widehat{\psi} \cdot p \cdot q \cdot A = q \cdot \widehat{\psi} \cdot (A + B) = q \cdot \widehat{\psi} \cdot A + q \cdot \widehat{\psi} \cdot B$ for some $B \in \ker q$. Hence, as $\widehat{\psi}(\ker q) = \ker q$, it follows that $q \cdot \widehat{\psi} \cdot p \cdot q \cdot A = q \cdot \widehat{\psi} \cdot A$ and so $\psi \circ q = q \circ \widehat{\psi}$. It remains to be shown that ψ is an automorphism. We have

$$\psi \cdot [A, B] - [\psi \cdot A, \psi \cdot B] = (q \circ \widehat{\psi}) \cdot (p \cdot [A, B] - [p \cdot A, p \cdot B])$$

for $A, B \in \mathfrak{g}$. However, $q \cdot (p \cdot [A, B] - [p \cdot A, p \cdot B]) = 0$ and so $p \cdot [A, B] - [p \cdot A, p \cdot B] \in \ker q$. Thus $\widehat{\psi} \cdot (p \cdot [A, B] - [p \cdot A, p \cdot B]) \in \ker q$. Therefore $(q \circ \widehat{\psi}) \cdot (p \cdot [A, B] - [p \cdot A, p \cdot B]) = 0$. Hence ψ is a Lie algebra homomorphism. Moreover

$$\begin{aligned} (q \circ \widehat{\psi} \circ p)(A) = 0 \iff (\widehat{\psi} \circ p)(A) \in \ker q \iff p(A) \in \ker q \\ \iff (q \circ p)(A) = 0 \iff A = 0. \end{aligned}$$

Therefore $\ker \psi = \{0\}$ and hence $\psi \in \mathsf{Aut}(\mathfrak{g})$.

Conversely, suppose there exists $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $q \circ \widehat{\psi} = \psi \circ q$. Let $A \in \ker q$. We have $(q \circ \widehat{\psi})(A) = (\psi \circ q)(A) = \psi(0) = 0$. Hence $\widehat{\psi}(A) \in \ker q$. Consequently, $\widehat{\psi}(\ker q) = \ker q$. ∎

COROLLARY 4.3. *For every* $\widehat{\psi} \in \mathsf{Aut}(\widehat{\mathfrak{g}})$, *there exists* $\psi \in \mathsf{Aut}(\mathfrak{g})$ *such that* $q \circ \widehat{\psi} = \psi \circ q$ *if and only if* $\ker q$ *is a fully characteristic ideal of* $\widehat{\mathfrak{g}}$.

COROLLARY 4.4. *Suppose* $\ker q$ *is a fully characteristic ideal of* $\widehat{\mathfrak{g}}$. *If* Γ_1 *and* Γ_2 *are equivalent, then* $q(\Gamma_1)$ *and* $q(\Gamma_2)$ *are equivalent.* (*Equivalently, if* $q(\Gamma_1)$ *and* $q(\Gamma_2)$ *are not equivalent, then neither are* Γ_1 *and* Γ_2.)

We now show that one can project classifications of subspaces, subalgebras, ideals, and full-rank subspaces; the subsequent theorem deals with lifting classifications.

THEOREM 4.5. *Suppose* $\ker q$ *is a fully characteristic ideal of* $\widehat{\mathfrak{g}}$. *Further, suppose* $\Gamma_i,\ i \in I$ *is a complete enumeration of class representatives for subspaces (resp. subalgebras, ideals, full-rank subspaces) of* $\widehat{\mathfrak{g}}$. *Then* $q(\Gamma_i)$, $i \in I$ *is a complete enumeration of class representatives for subspaces (resp. subalgebras, ideals, full-rank subspaces) of* $\mathfrak{g}$.

Proof. Let Γ be a subspace of $\mathfrak{g}$. Then $q^{-1}(\Gamma)$ is a subspace of $\widehat{\mathfrak{g}}$. Hence, as $\Gamma_i,\ i \in I$ is complete, there exists $i \in I$ such that $q^{-1}(\Gamma)$ is equivalent to Γ_i. Consequently, by Corollary 4.4, Γ is equivalent to $q(\Gamma_i)$. The same argument holds when Γ is a subalgebra, ideal, or full-rank subspace. ∎

Remark 4.6. The enumeration $q(\Gamma_i),\ i \in I$ may have redundancies even if $\Gamma_i,\ i \in I$ is nonredundant.

THEOREM 4.7. *Suppose* $\mathsf{Aut}(\mathfrak{g}) \circ q \subseteq q \circ \mathsf{Aut}(\widehat{\mathfrak{g}})$. *Further, suppose* Γ_i, $i \in I$ *is a complete enumeration of class representatives for subspaces (resp. subalgebras, ideals, full-rank subspaces) of* $\mathfrak{g}$. *Then for any subspace (resp. subalgebra, ideal, full-rank subspace)* Γ *of* $\widehat{\mathfrak{g}}$ *there exists* $i \in I$ *such that* Γ *is equivalent to a subspace* Γ' *of* $q^{-1}(\Gamma_i)$ *satisfying* $q(\Gamma') = \Gamma_i$.

Proof. Let Γ be a subspace of $\widehat{\mathfrak{g}}$. We have that $q(\Gamma)$ is a subspace of $\mathfrak{g}$ and so there exist $i \in I$ and an automorphism $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot q(\Gamma) = \Gamma_i$. As $\mathsf{Aut}(\mathfrak{g}) \circ q \subseteq q \circ \mathsf{Aut}(\widehat{\mathfrak{g}})$, there exists $\widehat{\psi} \in \mathsf{Aut}(\widehat{\mathfrak{g}})$ such that $\psi \circ q = q \circ \widehat{\psi}$. Thus $q \cdot \widehat{\psi}(\Gamma) = \Gamma_i$ and so $\widehat{\psi}(\Gamma) \subseteq q^{-1}(\Gamma_i)$. Accordingly, Γ is equivalent to a subspace $\Gamma' = \widehat{\psi}(\Gamma)$ of $q^{-1}(\Gamma_i)$ which satisfies $q(\Gamma') = \Gamma_i$. ∎

We collect the fully characteristic four-dimensional extensions of each three-dimensional Lie algebra in Table 1. Each three-dimensional Lie algebra has a four-dimensional fully characteristic extension. Hence, the classification of subspaces (resp. subalgebras, ideals, full-rank subspaces) of the thee-dimensional Lie algebras may readily be reobtained from the classification obtained in this paper. A few illustrative examples follow.

Table 1: Complete list of 4D fully characteristic extensions of 3D algebras

3D algebra	Fully characteristic 4D extensions
$3\mathfrak{g}_1$	$\mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1, \quad \mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$
$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$	$\mathfrak{g}_{3.2} \oplus \mathfrak{g}_1, \quad \mathfrak{g}_{3.4}^{\alpha} \oplus \mathfrak{g}_1, \quad \mathfrak{g}_{4.8}^{0}$
$\mathfrak{g}_{3.1}$	$\mathfrak{g}_{4.1}$
$\mathfrak{g}_{3.2}$	$\mathfrak{g}_{3.2} \oplus \mathfrak{g}_1, \quad \underset{\alpha\neq 1}{\mathfrak{g}_{4.2}^{\alpha}}, \quad \mathfrak{g}_{4.4}, \quad \mathfrak{g}_{3.7}, \quad \mathfrak{g}_{4.8}^{1}$
$\mathfrak{g}_{3.3}$	$\mathfrak{g}_{3.3} \oplus \mathfrak{g}_1, \quad \mathfrak{g}_{4.2}^{1}, \quad \underset{\alpha\neq 1}{\mathfrak{g}_{4.5}^{\alpha,\alpha}}$
$\mathfrak{g}_{3.4}^{0}$	$\mathfrak{g}_{3.4}^{0} \oplus \mathfrak{g}_1, \quad \mathfrak{g}_{4.8}^{-1}$
$\mathfrak{g}_{3.4}^{\alpha}$	$\mathfrak{g}_{3.4}^{\alpha} \oplus \mathfrak{g}_1, \mathfrak{g}_{4.2}^{\bar{\alpha}}, \quad \mathfrak{g}_{4.5}^{\bar{\alpha},\bar{\beta}}, \quad \mathfrak{g}_{4.5}^{\bar{\alpha},1}, \quad \underset{\bar{\alpha}\neq 0, \bar{\alpha}\neq 1}{\mathfrak{g}_{4.8}^{\bar{\alpha}}}$
$\mathfrak{g}_{3.5}^{0}$	$\mathfrak{g}_{3.5}^{0} \oplus \mathfrak{g}_1, \quad \mathfrak{g}_{4.9}^{0}$
$\mathfrak{g}_{3.5}^{\alpha}$	$\mathfrak{g}_{3.5}^{\alpha} \oplus \mathfrak{g}_1, \quad \mathfrak{g}_{4.6}^{\bar{\alpha},\bar{\beta}}, \quad \mathfrak{g}_{4.9}^{\alpha}$
$\mathfrak{g}_{3.6}$	$\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$
$\mathfrak{g}_{3.7}$	$\mathfrak{g}_{3.7} \oplus \mathfrak{g}_1$

EXAMPLE 4.8. The Lie algebra $\mathfrak{g}_{4.9}^{0}$ is a fully characteristic (central) extension of the Euclidean algebra $\mathfrak{se}\,(2) = \mathfrak{g}_{3.5}^{0}$. Indeed, the mapping

$$q : \mathfrak{g}_{4.9}^{0} \to \mathfrak{g}_{3.5}^{0}, \qquad \begin{bmatrix} 0 & -x & y & -2w \\ 0 & 0 & z & y \\ 0 & -z & 0 & x \\ 0 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & -z \\ y & z & 0 \end{bmatrix} = x\tilde{E}_1 + y\tilde{E}_2 + z\tilde{E}_3$$

is an epimorphism with $\ker q = \langle E_1 \rangle = \mathrm{Z}(\mathfrak{g}_{4.9}^{0})$. Any subalgebra of $\mathfrak{g}_{3.9}^{0}$ is equivalent to exactly one of the subalgebras:

$$\{0\}, \quad \langle E_1 \rangle, \quad \langle E_2 \rangle, \quad \langle E_4 \rangle, \quad \langle E_1, E_2 \rangle, \quad \langle E_1, E_4 \rangle, \quad \langle E_1, E_2, E_3 \rangle, \quad \mathfrak{g}_{4.9}^{0}.$$

Consequently, any subalgebra of $\mathfrak{g}_{3.5}^{0}$ is equivalent to at least one of the following subalgebras:

$$\{0\}, \quad \{0\}, \quad \langle \tilde{E}_1 \rangle, \quad \langle \tilde{E}_3 \rangle, \quad \langle \tilde{E}_1 \rangle, \quad \langle \tilde{E}_3 \rangle, \quad \langle \tilde{E}_1, \tilde{E}_2 \rangle, \quad \mathfrak{g}_{3.5}^{0}.$$

Once one has verified that $\langle \tilde{E}_1 \rangle$ and $\langle \tilde{E}_3 \rangle$ are not equivalent, one then has that any subalgebra of $\mathfrak{g}_{3.5}^{0}$ is equivalent to *exactly* one of the following subalgebras:

$$\{0\}, \quad \langle \tilde{E}_1 \rangle, \quad \langle \tilde{E}_3 \rangle, \quad \langle \tilde{E}_1, \tilde{E}_2 \rangle, \quad \mathfrak{g}_{3.5}^{0}.$$

Example 4.9. We consider again the fully characteristic extension $q : \mathfrak{g}_{4.9}^0 \to \mathfrak{g}_{3.5}^0$ of the Euclidean Lie algebra. Any ideal of $\mathfrak{g}_{3.5}^0$ is equivalent to exactly one of the following ideals:

$$\{0\}, \quad \langle \tilde{E}_1, \tilde{E}_2 \rangle, \quad \mathfrak{g}_{3.5}^0.$$

Hence any ideal of $\mathfrak{g}_{4.9}^0$ is equivalent to an ideal Γ' of one of the following ideals

$$\langle E_1 \rangle\,, \quad \langle E_1, E_2, E_3 \rangle\,, \quad \mathfrak{g}_{4.9}^0$$

satisfying $q(\Gamma') = \{0\}$, $q(\Gamma') = \langle \tilde{E}_1, \tilde{E}_2 \rangle$, or $q(\Gamma') = \mathfrak{g}_{3.5}^0$, respectively. Indeed, it turns out that these are the only ideals (apart from the trivial one) of $\mathfrak{g}_{4.9}^0$.

Example 4.10. The Lie algebra $\mathfrak{g}_{4.2}^1$ is a fully characteristic extension of the Lie algebra $\mathfrak{g}_{3.3}$. Indeed,

$$q : \mathfrak{g}_{4.2}^1 \to \mathfrak{g}_{3.3}, \qquad \begin{bmatrix} -z & 0 & 0 & w \\ 0 & -z & -z & x \\ 0 & 0 & -z & y \\ 0 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ y & -z & 0 \\ w & 0 & -z \end{bmatrix}$$

is an epimorphism with $\ker q = \langle E_2 \rangle$. Any proper full-rank subspace of $\mathfrak{g}_{4.2}^1$ is equivalent to $\langle E_1, E_2, E_4 \rangle$. We have $q(\langle E_1, E_2, E_4 \rangle) = \mathfrak{g}_{3.3}$. Hence, $\mathfrak{g}_{3.3}$ has no proper full-rank subspaces.

Example 4.11. The Lie algebra $\mathfrak{g}_{4.8}^{-1}$ is a fully characteristic (central) extension of the semi-Euclidean algebra $\mathfrak{g}_{3.4}^0 = \mathfrak{se}\,(1,1)$. Indeed, the mapping $q : \mathfrak{g}_{4.8}^{-1} \to \mathfrak{g}_{3.4}^0$,

$$\begin{bmatrix} 0 & x & w \\ 0 & z & y \\ 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ x+y & 0 & -z \\ x-y & -z & 0 \end{bmatrix} = (x+y)\tilde{E}_1 + (x-y)\tilde{E}_2 + z\tilde{E}_3$$

is an epimorphism with $\ker q = \langle E_1 \rangle = \mathrm{Z}(\mathfrak{g}_{4.8}^{-1})$. Any full-rank subspace of $\mathfrak{g}_{4.8}^{-1}$ is equivalent to exactly one of the following subspaces:

$$\langle E_2 + E_3, E_4 \rangle\,, \quad \langle E_2, E_3, E_4 \rangle\,, \quad \langle E_1, E_2 + E_3, E_4 \rangle\,, \quad \mathfrak{g}_{3.8}^{-1}.$$

Hence any full-rank subspace of $\mathfrak{g}_{3.4}^{-1}$ is equivalent to at least one of the following subspaces:

$$\langle \tilde{E}_1, \tilde{E}_3 \rangle, \quad \mathfrak{g}_{3.4}^0, \quad \langle \tilde{E}_1, \tilde{E}_3 \rangle, \quad \mathfrak{g}_{3.4}^0.$$

Consequently, any proper full-rank subspace of $\mathfrak{g}_{3.4}^{-1}$ is equivalent to $\langle \tilde{E}_1, \tilde{E}_3 \rangle$.

5. Final remarks

A classification of subalgebras (but not all subspaces) of four-dimensional Lie algebras was obtained in [18], up to inner automorphism. This classification has been employed by several authors (especially in the field of mathematical physics, see e.g., [12, 15, 19, 20, 21]). For instance, in [19] the (fully characteristic) ideals are identified among the subalgebras; these ideals are then used in finding a complete set of inequivalent realizations of real Lie algebras (of dimension no greater than four).

It turns out that equivalence up to automorphism (as studied in this paper) is considerably weaker than equivalence up to inner automorphism. The main reason for our interest in the classification of subspaces is in connection with geometric control and sub-Riemannian structures on Lie groups. More precisely, the classification of full-rank subspaces of a Lie algebra yields a classification of the invariant (bracket-generating) distributions or the homogeneous invariant control affine systems on the corresponding simply connected Lie group (cf. [7, 2, 8, 3]).

A. Classification of low-dimensional Lie algebras

The classification of three- and four-dimensional (real) Lie algebras is well known (see, e.g., [14], [19], and the references therein). We prefer to use (a modified version of) the enumeration of these Lie algebras due to Mubarakzyanov ([16]), similar to that used by Patera et al. ([18, 17]), which is complete and nonredundant. However, in the three-dimensional case, we use the commutator relations in the Bianchi-Behr form ([13]).

A.1. Three-dimensional Lie algebras.

In terms of an (appropriate) ordered basis (E_1, E_2, E_3), the commutator operation is given by

$$\begin{aligned}
[E_2, E_3] &= n_1 E_1 - \alpha\, E_2 \\
[E_3, E_1] &= \alpha\, E_1 + n_2 E_2 \\
[E_1, E_2] &= n_3 E_3.
\end{aligned}$$

The (Bianchi-Behr) structure parameters α, n_1, n_2, n_3 for each type are given in Table 2.

Table 2: Bianchi-Behr classification of 3D Lie algebras

Type	Bianchi	α	n_1	n_2	n_3	Unimodular	Nilpotent	Compl. Solv.	Exponential	Solvable	Simple	Representatives
$3\mathfrak{g}_1$	I	0	0	0	0	•	•	•	•	•		$\mathbb{R}^3$
$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$	III	1	1	-1	0			•	•	•		$\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$, $\mathfrak{g}_{3.4}^1$
$\mathfrak{g}_{3.1}$	II	0	1	0	0	•	•	•	•	•		$\mathfrak{h}_3$
$\mathfrak{g}_{3.2}$	IV	1	1	0	0			•	•	•		
$\mathfrak{g}_{3.3}$	V	1	0	0	0			•	•	•		
$\mathfrak{g}_{3.4}^0$	VI_0	0	1	-1	0	•		•	•	•		$\mathfrak{se}(1,1)$
$\mathfrak{g}_{3.4}^\alpha$	VI_α	$\alpha>0$ $\alpha\neq 1$	1	-1	0			•	•	•		
$\mathfrak{g}_{3.5}^0$	VII_0	0	1	1	0	•				•		$\mathfrak{se}(2)$
$\mathfrak{g}_{3.5}^\alpha$	VII_α	$\alpha>0$	1	1	0				•	•		
$\mathfrak{g}_{3.6}$	$VIII$	0	1	1	-1	•					•	$\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{so}(2,1)$
$\mathfrak{g}_{3.7}$	IX	0	1	1	1	•					•	$\mathfrak{su}(2)$, $\mathfrak{so}(3)$

A.2. Four-dimensional Lie algebras. We distinguish between the decomposable (as direct sums of lower-dimensional Lie algebras) and indecomposable algebras. There are twelve types of decomposable algebras (in fact, ten algebras and two one-parameter families of algebras) and twelve types of indecomposable algebras (in fact, seven algebras, three one-parameter families of algebras, and two two-parameter families of algebras). In terms of an (appropriate) ordered basis (E_1, E_2, E_3, E_4), the commutator relations for each four-dimensional Lie algebra are given in Table 3.

We collect some basic properties for each algebra in Table 4. For each algebra $\mathfrak{g}$, the quotient $\mathfrak{g}/\,\mathrm{Z}(\mathfrak{g})$ is displayed when $\mathrm{Z}(\mathfrak{g})$ is nontrivial. We also list all fully characteristic ideals of codimension one. Furthermore, we indicate those algebras that admit an invariant scalar product (abbreviated ISP), i.e., a nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ satisfying $\langle A, [B, C] \rangle = \langle [A, B], C \rangle$ for all $A, B, C \in \mathfrak{g}$.

Table 3: Four-dimensional Lie algebras (commutator relations)

Type	Non-zero commutators			Parameter
$4\mathfrak{g}_1$				
$\mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1$	$[E_1, E_2] = E_1$			
$2\mathfrak{g}_{2.1}$	$[E_1, E_2] = E_1$	$[E_3, E_4] = E_3$		
$\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$	$[E_2, E_3] = E_1$			
$\mathfrak{g}_{3.2} \oplus \mathfrak{g}_1$	$[E_2, E_3] = E_1 - E_2$	$[E_3, E_1] = E_1$		
$\mathfrak{g}_{3.3} \oplus \mathfrak{g}_1$	$[E_2, E_3] = -E_2$	$[E_3, E_1] = E_1$		
$\mathfrak{g}_{3.4}^0 \oplus \mathfrak{g}_1$	$[E_2, E_3] = E_1$	$[E_3, E_1] = -E_2$		
$\mathfrak{g}_{3.4}^\alpha \oplus \mathfrak{g}_1$	$[E_2, E_3] = E_1 - \alpha E_2$	$[E_3, E_1] = \alpha E_1 - E_2$		$\alpha > 0$, $\alpha \neq 1$
$\mathfrak{g}_{3.5}^0 \oplus \mathfrak{g}_1$	$[E_2, E_3] = E_1$	$[E_3, E_1] = E_2$		
$\mathfrak{g}_{3.5}^\alpha \oplus \mathfrak{g}_1$	$[E_2, E_3] = E_1 - \alpha E_2$	$[E_3, E_1] = \alpha E_1 + E_2$		$\alpha > 0$
$\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$	$[E_2, E_3] = E_1$	$[E_3, E_1] = E_2$	$[E_1, E_2] = -E_3$	
$\mathfrak{g}_{3.7} \oplus \mathfrak{g}_1$	$[E_2, E_3] = E_1$	$[E_3, E_1] = E_2$	$[E_1, E_2] = E_3$	

Table 3: (continued)

Type	Non-zero commutators			Parameter
$\mathfrak{g}_{4.1}$	$[E_2, E_4] = E_1$	$[E_3, E_4] = E_2$		
$\mathfrak{g}_{4.2}^{\alpha}$	$[E_1, E_4] = \alpha\, E_1$	$[E_2, E_4] = E_2$	$[E_3, E_4] = E_2 + E_3$	$\alpha \neq 0$
$\mathfrak{g}_{4.3}$	$[E_1, E_4] = E_1$	$[E_3, E_4] = E_2$		
$\mathfrak{g}_{4.4}$	$[E_1, E_4] = E_1$	$[E_2, E_4] = E_1 + E_2$	$[E_3, E_4] = E_2 + E_3$	
$\mathfrak{g}_{4.5}^{\alpha,\beta}$	$[E_1, E_4] = E_1$	$[E_2, E_4] = \beta\, E_2$	$[E_3, E_4] = \alpha\, E_3$	$-1<\alpha\leq\beta\leq 1$, $\alpha\beta\neq 0$ or $\alpha=-1, 0<\beta\leq 1$
$\mathfrak{g}_{4.6}^{\alpha,\beta}$	$[E_1, E_4] = \alpha\, E_1$	$[E_2, E_4] = \beta\, E_2 - E_3$	$[E_3, E_4] = E_2 + \beta\, E_3$	$\alpha > 0,\ \beta \in \mathbb{R}$
$\mathfrak{g}_{4.7}$	$[E_1, E_4] = 2E_1$ $[E_2, E_3] = E_1$	$[E_2, E_4] = E_2$	$[E_3, E_4] = E_2 + E_3$	
$\mathfrak{g}_{4.8}^{-1}$	$[E_2, E_3] = E_1$	$[E_2, E_4] = E_2$	$[E_3, E_4] = -E_3$	
$\mathfrak{g}_{4.8}^{\alpha}$	$[E_1, E_4] = (1+\alpha)\, E_1$ $[E_2, E_3] = E_1$	$[E_2, E_4] = E_2$	$[E_3, E_4] = \alpha\, E_3$	$-1 < \alpha \leq 1$
$\mathfrak{g}_{4.9}^{0}$	$[E_2, E_3] = E_1$	$[E_2, E_4] = -E_3$	$[E_3, E_4] = E_2$	
$\mathfrak{g}_{4.9}^{\alpha}$	$[E_1, E_4] = 2\alpha\, E_1$ $[E_2, E_3] = E_1$	$[E_2, E_4] = \alpha\, E_2 - E_3$	$[E_3, E_4] = E_2 + \alpha\, E_3$	$\alpha > 0$
$\mathfrak{g}_{4.10}$	$[E_1, E_3] = E_1$ $[E_2, E_4] = E_1$	$[E_2, E_3] = E_2$	$[E_1, E_4] = -E_2$	

Table 4: Four-dimensional Lie algebras (properties)

Type	Fully char. 3D ideal	$\mathfrak{g}/Z(\mathfrak{g})$	Unimodular	Nilpotent	Compl. Solv.	Exponential	Solvable	Admits ISP
$4\mathfrak{g}_1$	—	$\{0\}$	•	•	•	•	•	•
$\mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1$	$3\mathfrak{g}_1$	$\mathfrak{g}_{2.1}$			•	•	•	
$2\mathfrak{g}_{2.1}$	$\mathfrak{g}_{3.3}$, $\mathfrak{g}_{3.4}^0$	—			•	•	•	
$\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$	—	$2\mathfrak{g}_1$	•	•	•	•	•	
$\mathfrak{g}_{3.2} \oplus \mathfrak{g}_1$	$3\mathfrak{g}_1$	$\mathfrak{g}_{3.2}$			•	•	•	
$\mathfrak{g}_{3.3} \oplus \mathfrak{g}_1$	$3\mathfrak{g}_1$	$\mathfrak{g}_{3.3}$			•	•	•	
$\mathfrak{g}_{3.4}^0 \oplus \mathfrak{g}_1$	$3\mathfrak{g}_1$	$\mathfrak{g}_{3.4}^0$	•		•	•	•	
$\mathfrak{g}_{3.4}^\alpha \oplus \mathfrak{g}_1$	$3\mathfrak{g}_1$	$\mathfrak{g}_{3.4}^\alpha$			•	•	•	
$\mathfrak{g}_{3.5}^0 \oplus \mathfrak{g}_1$	$3\mathfrak{g}_1$	$\mathfrak{g}_{3.5}^0$	•				•	
$\mathfrak{g}_{3.5}^\alpha \oplus \mathfrak{g}_1$	$3\mathfrak{g}_1$	$\mathfrak{g}_{3.5}^\alpha$				•	•	
$\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$	$\mathfrak{g}_{3.6}$	$\mathfrak{g}_{3.6}$	•					•
$\mathfrak{g}_{3.7} \oplus \mathfrak{g}_1$	$\mathfrak{g}_{3.7}$	$\mathfrak{g}_{3.7}$	•					•
$\mathfrak{g}_{4.1}$	$3\mathfrak{g}_1$	$\mathfrak{g}_{3.1}$	•	•	•	•	•	
$\mathfrak{g}_{4.2}^\alpha$	$3\mathfrak{g}_1$	—	$\alpha{=}-2$		•	•	•	
$\mathfrak{g}_{4.3}$	$3\mathfrak{g}_1$	$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$			•	•	•	
$\mathfrak{g}_{4.4}$	$3\mathfrak{g}_1$	—			•	•	•	
$\mathfrak{g}_{4.5}^{\alpha,\beta}$	$3\mathfrak{g}_1$	—	$\alpha{+}\beta{=}-1$		•	•	•	
$\mathfrak{g}_{4.6}^{\alpha,\beta}$	$3\mathfrak{g}_1$	—	$\alpha{=}-2\beta$			$\beta{\neq}0$	•	
$\mathfrak{g}_{4.7}$	$\mathfrak{g}_{3.1}$	—			•	•	•	
$\mathfrak{g}_{4.8}^{-1}$	$\mathfrak{g}_{3.1}$	$\mathfrak{g}_{3.4}^0$	•		•	•	•	•
$\mathfrak{g}_{4.8}^\alpha$	$\mathfrak{g}_{3.1}$, ${\mathfrak{g}_{3.3}}_{\alpha=0}$	—			•	•	•	
$\mathfrak{g}_{4.9}^0$	$\mathfrak{g}_{3.1}$	$\mathfrak{g}_{3.5}^0$	•				•	•
$\mathfrak{g}_{4.9}^\alpha$	$\mathfrak{g}_{3.1}$	—				•	•	
$\mathfrak{g}_{4.10}$	$\mathfrak{g}_{3.3}$, $\mathfrak{g}_{3.5}^0$	—					•	

B. Classification procedure and proofs

The classification procedure is described in Appendix B.1. Details for the classification of a typical case is given in Appendix B.2. In Appendix B.3 and Appendix B.4 proofs are provided for the classification of subspaces of $\mathfrak{g}_{3.6}\oplus\mathfrak{g}_1$ and $\mathfrak{g}_{3.7}\oplus\mathfrak{g}_2$ where the usual verification procedure (using a computer algebra system) breaks down.

B.1. Subspace classification. The classification procedure for each Lie algebra $\mathfrak{g}$ is as follows. First, a standard computation yields the automorphism group $\mathsf{Aut}(\mathfrak{g})$ (see, e.g., [19, 10]). One then constructs class representatives by considering the action of automorphisms on a typical subspace. Finally, one verifies that none of these representatives are equivalent. This procedure has been successfully applied in classifying certain classes of (affine) subspaces of three-dimensional Lie algebras ([7], see also [4, 5, 6]) as well as some higher-dimensional Lie algebras ([9, 1, 2]). In the four-dimensional case, we verify nonredundancy and completeness of the classification by using a computer algebra system (Mathematica).

Note B.1. For the sake of simplicity, we shall discuss here only the case when the enumeration of subspaces is finite and the Lie algebra is fixed. It is not difficult to adapt the approach for the case of an (infinite) parametrized family of Lie algebras, or the case where the prospective enumeration contains (infinite) parametrized families of subspaces.

Finding a prospective (finite) enumeration $\Gamma_1, \ldots, \Gamma_n$ of subspaces is not difficult; we provide details for Theorem 2.1 in Appendix B.2. The problem then reduces to verifying that (a) the enumeration is *nonredundent*, i.e., no two subspace Γ_i and Γ_j are equivalent, and that (b) the enumeration is *complete*, i.e., any subspace is equivalent to at least one subspace Γ_i. We can apply simple (although computationally intensive) algorithms to verify (a) and (b); these algorithms are described bellow.

Note B.2. For the Lie algebras $\mathfrak{g}_{3.6}\oplus\mathfrak{g}_1$ and $\mathfrak{g}_{3.7}\oplus\mathfrak{g}_1$ such a computer aided verification does not work. (Due to the nature of the automorphism groups, these algorithms become impractical to implement.) In these cases an approach similar to that used in [4] is implemented; proofs are appended.

Any subspace Γ has a basis $B_1, \ldots, B_\ell$; we write this basis as a matrix $\mathbf{B} = \begin{bmatrix} B_1 & \cdots & B_\ell \end{bmatrix}$. (Here each B_i is identified with its corresponding coordinate column vector.) By a slight abuse of notation, we write $\Gamma = \langle \mathbf{B} \rangle = \langle B_1, \ldots, B_\ell \rangle$. Two bases $\mathbf{B}$ and $\mathbf{B}'$ define the same subspace exactly when there exists $R \in \mathsf{GL}\,(\ell, \mathbb{R})$ such that $\mathbf{B} = \mathbf{B}'\, R$. Consequently, two subspaces $\langle \mathbf{B} \rangle$ and $\langle \mathbf{B}' \rangle$ are equivalent exactly when there exist $\psi \in \mathsf{Aut}(\mathfrak{g})$ and $R \in \mathsf{GL}\,(\ell, \mathbb{R})$ such that $\psi\, \mathbf{B} = \mathbf{B}'\, R$. (Throughout, each automorphisms ψ is identified with its matrix.)

Nonredundancy. Given a prospective enumeration $\langle \mathbf{B}_1 \rangle, \ldots, \langle \mathbf{B}_n \rangle$ of the ℓ-dimensional subspaces (for a Lie algebra $\mathfrak{g}$), we wish to show that no two subspaces $\langle \mathbf{B}_i \rangle$ and $\langle \mathbf{B}_j \rangle$ are equivalent. Formally, this is equivalent to showing that the statement

$$\bigvee_{1 \leq i < j \leq n} \exists_{\psi \in \mathsf{Aut}(\mathfrak{g})}\, \exists_{R \in \mathsf{GL}\,(\ell, \mathbb{R})} \quad \psi\, \mathbf{B}_i = \mathbf{B}_j\, R \tag{B.1}$$

is false. (Here $\vee$ denotes logical disjunction.) Given the automorphism group $\mathsf{Aut}(\mathfrak{g})$ as a parametrized matrix Lie group, the truth value of (B.1) can fairly easily be determined by using a computer algebra system.

Completeness. Given a prospective enumeration $\langle \mathbf{B}_1 \rangle, \ldots, \langle \mathbf{B}_n \rangle$ of the ℓ-dimensional subspaces, we wish to show that any subspace $\langle \mathbf{B} \rangle$ is equivalent to at least one subspace $\langle \mathbf{B}_i \rangle$. This will be the case exactly when the statement

$$\forall_{\mathbf{B} \in \mathbb{R}^{\dim \mathfrak{g} \times \ell},\ \det(\mathbf{B}^\top \mathbf{B}) \neq 0}\ \exists_{i \in \{1, \ldots, n\}}\ \exists_{\psi \in \mathsf{Aut}(\mathfrak{g})}\ \exists_{R \in \mathsf{GL}\,(\ell, \mathbb{R})} \quad \psi\, \mathbf{B} = \mathbf{B}_i\, R \tag{B.2}$$

is true. However, in our experience (B.2) cannot be evaluated (or rather, the evaluation does not terminate) in Mathematica, except in the one-dimensional case. Hence, we express (B.2) in a more computationally amenable form.

As $\langle \mathbf{B} \rangle = \langle \mathbf{B}\, R \rangle$ for any $R \in \mathsf{GL}\,(\ell, \mathbb{R})$, we can reduce the collection of possible bases $\mathbf{B} \in \mathbb{R}^{\dim \mathfrak{g} \times \ell}$, $\det(\mathbf{B}^\top \mathbf{B}) \neq 0$ for the ℓ-dimensional subspaces. Henceforth, we shall assume that $\dim \mathfrak{g} = 4$.

Lemma B.3. *Any two-dimensional subspace admits a basis* $\mathbf{B} \in \mathcal{B}_2$, *where*

$$\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ s_1 & s_2 \\ s_3 & s_4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ s_1 & s_2 \\ 0 & 1 \\ s_3 & s_4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ s_1 & s_2 \\ s_3 & s_4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} s_1 & s_2 \\ 1 & 0 \\ 0 & 1 \\ s_3 & s_4 \end{bmatrix}, \begin{bmatrix} s_1 & s_2 \\ 1 & 0 \\ s_3 & s_4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right.$$
$$\left. : s_1, s_2, s_3, s_4 \in \mathbb{R} \right\}.$$

Proof. Let

$$\mathbf{B} = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \\ s_5 & s_6 \\ s_7 & s_8 \end{bmatrix}.$$

We have that exactly two rows of $\mathbf{B}$ are linearly independent (as $\langle \mathbf{B} \rangle$ is two-dimensional). Suppose the first two rows are linearly independent, i.e., $s_1 s_4 - s_2 s_3 \neq 0$. Then

$$\mathbf{B}' = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \\ s_5 & s_6 \\ s_7 & s_8 \end{bmatrix} \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ s_5' & s_6' \\ s_7' & s_8' \end{bmatrix}$$

for some $s_5', s_6', s_7', s_8' \in \mathbb{R}$. Moreover $\langle \mathbf{B} \rangle = \langle \mathbf{B}' \rangle$. The other possible bases correspond to other combinations of rows of $\mathbf{B}$ being linearly independent. ∎

Accordingly, a prospective enumeration $\langle \mathbf{B}_1 \rangle, \ldots, \langle \mathbf{B}_n \rangle$ of the two-dimensional subspaces of $\mathfrak{g}$ is complete if and only if the statement

$$\forall_{\mathbf{B} \in \mathcal{B}_2} \, \exists_{i \in \{1,\ldots,n\}} \, \exists_{\psi \in \mathsf{Aut}(\mathfrak{g})} \, \exists_{R \in \mathsf{GL}\,(\ell,\mathbb{R})} \quad \psi \, \mathbf{B} = \mathbf{B}_i \, R \tag{B.3}$$

is true. Likewise, for the three-dimensional case we have the following reduced collection of bases.

Lemma B.4. *Any three-dimensional subspace admits a basis* $\mathbf{B} \in \mathcal{B}_3$, *where*

$$\mathcal{B}_3 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ s_1 & s_2 & s_3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s_1 & s_2 & s_3 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ s_1 & s_2 & s_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} s_1 & s_2 & s_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : s_1, s_2, s_3 \in \mathbb{R} \right\}.$$

Hence, a prospective enumeration $\langle \mathbf{B}_1 \rangle, \ldots, \langle \mathbf{B}_n \rangle$ of the three-dimensional subspaces of $\mathfrak{g}$ is complete if and only if the statement

$$\forall_{\mathbf{B} \in \mathcal{B}_3} \, \exists_{i \in \{1,\ldots,n\}} \, \exists_{\psi \in \mathsf{Aut}(\mathfrak{g})} \, \exists_{R \in \mathsf{GL}\,(\ell,\mathbb{R})} \quad \psi \, \mathbf{B} = \mathbf{B}_i \, R \tag{B.4}$$

is true. In most cases, (B.3) and (B.4) can be evaluated using a computer algebra system. However, there are a number of exceptions in which we use different reduced collections of bases. For several algebras we used

$$\mathcal{B}_2' = \left\{ \begin{bmatrix} s_1 & s_2 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ s_1 & s_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ s_1 & s_2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ s_1 & s_2 \\ s_3 & s_4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} s_1 & s_2 \\ 1 & 0 \\ s_3 & s_4 \\ 0 & 1 \end{bmatrix} \right.$$
$$\left. : s_1, s_2, s_3, s_4 \in \mathbb{R} \right\}.$$

$$\mathcal{B}_3' = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ s_1 & s_2 & s_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s_1 & s_2 & s_3 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} s_1 & s_2 & s_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right.$$
$$\left. : s_1, s_2, s_3 \in \mathbb{R} \right\}.$$

(Here we separate bases for which the fourth row is zero from those for which it is not.) In a few cases, we used

$$\mathcal{B}_2'' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ s_1 & s_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ s_1 & s_2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} s_1 & s_2 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ s_1 & s_2 \\ 0 & 1 \\ s_3 & s_4 \end{bmatrix}, \begin{bmatrix} s_1 & s_2 \\ 1 & 0 \\ 0 & 1 \\ s_3 & s_4 \end{bmatrix}, \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right.$$
$$\left. : s_1, s_2, s_3, s_4 \in \mathbb{R} \right\}.$$

$$\mathcal{B}_3'' = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} s_1 & s_2 & s_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ s_1 & s_2 & s_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ s_1 & s_2 & s_3 \end{bmatrix} \right.$$
$$\left. : s_1, s_2, s_3 \in \mathbb{R} \right\}.$$

(Here we separate bases for which third row is zero from those for which it is not.) Finally, for $\mathfrak{g}_{4.10}$ we used the collection

$$\mathcal{B}_2''' = \left\{ \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & s_1 \\ 0 & s_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ s_1 & 0 \\ 0 & 1 \\ 0 & s_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ s_1 & 0 \\ 0 & s_2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} s_1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & s_2 \end{bmatrix}, \begin{bmatrix} s_1 & 0 \\ 1 & 0 \\ 0 & s_2 \\ 0 & 1 \end{bmatrix} \right.$$
$$\left. : s_1, s_2, s_3, s_4 \in \mathbb{R} \right\}.$$

in testing the completeness of the two-dimensional subspaces (and $\mathcal{B}_3'$ for the thee-dimensional subspaces). One can show that any two-dimensional subspace admits a basis $\mathbf{B} \in \mathcal{B}_2'''$ by considering whether or not the last two rows are linearly independent and then whether or not first two rows are linearly independent.

Subspace structure. Given a complete and nonredundent enumeration $\langle \mathbf{B}_1 \rangle, \ldots, \langle \mathbf{B}_n \rangle$ of the subspaces of $\mathfrak{g}$, we wish to determine exactly which subspaces are (Abelian or non-Abelian) subalgebras, are (noncharacteristic, characteristic, or fully characteristic) ideals, or have full rank. An ideal $\mathfrak{n}$ is *characteristic* if it is invariant under all derivations, i.e., $\psi \cdot \mathfrak{n} \subseteq \mathfrak{n}$ for $\psi \in \mathfrak{der}(\mathfrak{g})$. On the other hand, an ideal $\mathfrak{n}$ is *fully characteristic* if it is invariant under all automorphisms, i.e., $\psi \cdot \mathfrak{n} = \mathfrak{n}$ for $\psi \in \mathsf{Aut}(\mathfrak{g})$. A subspace Γ is said to have *full rank* if it generates the whole Lie algebra, i.e., the smallest Lie algebra $\mathsf{Lie}(\Gamma)$ containing Γ is $\mathfrak{g}$.

It is easy to determine which subspaces are (Abelian or non-Abelian) subalgebras and which are (noncharacteristic, characteristic, or fully characteristic) ideals. We have the following characterization of full-rank subspaces (for four-dimensional Lie algebras). No one-dimensional subspace has full rank. A two-dimensional subspace $\langle \mathbf{B} \rangle$, $\mathbf{B} = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$ has full rank exactly when the matrix

$$M = \begin{bmatrix} B_1 & B_2 & [B_1, B_2] & [B_1, [B_1, B_2]] & [B_2, [B_1, B_2]] \end{bmatrix}$$

has full rank, i.e., $\det(MM^\top) \neq 0$. Similarly, a three-dimensional subspace $\langle \mathbf{B} \rangle$, $\mathbf{B} = \begin{bmatrix} B_1 & B_2 & B_3 \end{bmatrix}$ has full rank exactly when the matrix

$$M = \begin{bmatrix} B_1 & B_2 & B_3 & [B_1, B_2] & [B_1, B_3] & [B_2, B_3] \end{bmatrix}$$

has full rank, i.e., $\det(MM^\top) \neq 0$.

B.2. Proof for Theorem 2.1 (algebra $\mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1$). We prove only the assertion that any proper subspace of $\mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1$ is equivalent to one of the subspaces listed. Let $\Gamma = \langle a_1E_1 + a_2E_2 + a_3E_3 + a_4E_4\rangle$ be a one-dimensional subspace of $\mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1$. Suppose $a_4 \neq 0$ or $a_3 \neq 0$. If $a_4 \neq 0$, then $\Gamma = \Gamma' = \langle a_1'E_1 + a_2'E_2 + a_3'E_3 + E_4\rangle$ for some $a_1', a_2', a_3' \in \mathbb{R}$ and if $a_4 = 0$, then

$$\psi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

is an automorphism such that $\Gamma' = \psi \cdot \Gamma = \langle a_1'E_1 + a_2'E_2 + E_4\rangle$ for some $a_1', a_2' \in \mathbb{R}$. In either case we have that Γ is equivalent to a subspace $\Gamma' = \langle a_1'E_1 + a_2'E_2 + a_3'E_3 + E_4\rangle$ for some $a_1', a_2', a_3' \in \mathbb{R}$. If $a_2' \neq 0$, then

$$\psi = \begin{bmatrix} 1 & -\frac{a_1'}{a_2'} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -a_3' \\ 0 & 1 & 0 & -a_2' \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma' = \psi \cdot \langle a_1'E_1 + a_2'E_2 + a_3'E_3 + E_4\rangle = \langle E_2\rangle$. If $a_2' = 0$ and $a_1' \neq 0$, then

$$\psi = \begin{bmatrix} \frac{1}{a_1'} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -a_3' \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma' = \psi \cdot \langle a_1'E_1 + a_3'E_3 + E_4\rangle = \langle E_1 + E_4\rangle$. If $a_1' = a_2' = 0$, then

$$\psi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -a_3' \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma' = \psi \cdot \langle a_3'E_3 + E_4\rangle = \langle E_4\rangle$. On the other hand, suppose $a_3 = a_4 = 0$. If $a_2 \neq 0$, then

$$\psi = \begin{bmatrix} 1 & -\frac{a_1}{a_2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma = \psi \cdot \langle a_1 E_1 + a_2 E_2 \rangle = \langle E_2 \rangle$. If $a_2 = 0$, then $\Gamma = \langle E_1 \rangle$.

Accordingly, any one-dimensional subspace of $\mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1$ is equivalent to $\langle E_1 \rangle$, $\langle E_2 \rangle$, $\langle E_4 \rangle$, or $\langle E_1 + E_4 \rangle$. Completeness and nonredundancy can now be verified as described in Appendix B.1.

Remark B.5. Alternatively, nonredundency can often be handled by identifying some basic invariants. For example, any automorphism ψ preserves $\langle E_1 \rangle$, i.e., $\psi \cdot \langle E_1 \rangle = \langle E_1 \rangle$. Thus $\langle E_1 \rangle$ is not equivalent to $\langle E_2 \rangle$, $\langle E_4 \rangle$, or $\langle E_1 + E_4 \rangle$.

Let $\Gamma = \langle \sum a_i E_i, \sum b_i E_i \rangle$ be a two-dimensional subspace of $\mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1$. Suppose $E^2(\Gamma) \neq \{0\}$, i.e., $a_2^2 + b_2^2 \neq 0$. (Here E^2 denotes the corresponding element of the dual basis.) We may assume that $a_2 = 1$ and $b_2 = 0$, i.e., $\Gamma = \langle a_1 E_1 + E_2 + a_3 E_3 + a_3 E_4, b_1 E_1 + b_3 E_3 + b_4 E_4 \rangle$. If $b_3 = b_4 = 0$, then $b_1 \neq 0$ and so

$$\begin{bmatrix} \frac{1}{b_1} & -\frac{a_1}{b_1} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -a_3 & 1 & 0 \\ 0 & -a_4 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma = \langle E_1, E_2 \rangle$. If $b_3^2 + b_4^2 \neq 0$, then

$$\psi = \begin{bmatrix} x & -a_1 x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{a_4 b_3 - a_3 b_4}{b_3^2 + b_4^2} & \frac{b_4}{b_3^2 + b_4^2} & -\frac{b_3}{b_3^2 + b_4^2} \\ 0 & -\frac{a_3 b_3 + a_4 b_4}{b_3^2 + b_4^2} & \frac{b_3}{b_3^2 + b_4^2} & \frac{b_4}{b_3^2 + b_4^2} \end{bmatrix}, \qquad x \neq 0$$

is an automorphism such that $\psi \cdot \Gamma = \langle E_2, x b_1 E_1 + E_4 \rangle$. Hence, Γ is equivalent to $\langle E_2, E_4 \rangle$ if $b_1 = 0$ and is equivalent to $\langle E_2, E_1 + E_4 \rangle$ if $b_1 \neq 0$ (in this case we take $x = \frac{1}{b_1}$). On the other hand, suppose $E^2(\Gamma) = \{0\}$, i.e., $\Gamma = \langle a_1 E_1 + a_3 E_3 + a_4 E_4, b_1 E_1 + b_3 E_3 + b_4 E_4 \rangle$. If $b_3 = a_3 b_3$, then

$$\psi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma = \langle a_1 E_1 + E_4, b_1 E_1 + b_4 E_4 \rangle = \langle E_1, E_4 \rangle$.

If $a_1 \neq 0$ and $b_3 - a_3b_4 \neq 0$, then

$$\psi = \begin{bmatrix} \frac{1}{a_1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{b_1 - a_1b_4}{a_1(b_3 - a_3b_4)} & \frac{-a_3b_1 + a_1b_3}{a_1(b_3 - a_3b_4)} \\ 0 & 0 & \frac{1}{b_3 - a_3b_4} & \frac{a_3}{-b_3 + a_3b_4} \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma = \left\langle E_1 + E_3, \frac{b_1}{a_1}E_1 + \frac{b_1}{a_1}E_3 + E_4 \right\rangle = \langle E_1 + E_3, E_4 \rangle$. If $a_1 = 0$ and $b_3 - a_3b_4 \neq 0$, then

$$\psi = \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{b_3 - a_3b_4} & \frac{a_3}{-b_3 + a_3b_4} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad x \neq 0$$

is an automorphism such that

$$\psi \cdot \Gamma = \langle E_4, xb_1E_1 + E_3 + b_4E_4 \rangle = \langle E_4, xb_1E_1 + E_3 \rangle \,.$$

Hence, Γ is equivalent to $\langle E_3, E_4 \rangle$ if $b_1 = 0$ and is equivalent to $\langle E_1 + E_3, E_4 \rangle$ if $b_1 \neq 0$ (in this case we take $x = \frac{1}{b_1}$).

Accordingly, any two-dimensional subspace is equivalent to $\langle E_1, E_2 \rangle$, $\langle E_1, E_4 \rangle$, $\langle E_2, E_4 \rangle$, $\langle E_3, E_4 \rangle$, $\langle E_1 + E_4, E_2 \rangle$, or $\langle E_1 + E_3, E_4 \rangle$. Completeness and nonredundancy can again be verified as described in Appendix B.1. As a typical example, we illustrate this computational approach to nonredundancy for one case. Suppose $\langle E_1, E_2 \rangle$ and $\langle E_1, E_4 \rangle$ are equivalent. Then there exists an automorphism

$$\psi = \begin{bmatrix} \sigma & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 & a_6 \end{bmatrix}$$

such that

$$\begin{bmatrix} \sigma & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x_1 & x_2 & x_3 \\ 0 & x_4 & x_5 & x_6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix}$$

for some $r_1, r_2, r_3, r_4 \in \mathbb{R}$, $r_1r_4 - r_2r_3 \neq 0$. That is,

$$\begin{bmatrix} x_1 & x_2 \\ 0 & 1 \\ 0 & x_3 \\ 0 & x_6 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 \\ 0 & 0 \\ 0 & 0 \\ r_3 & r_4 \end{bmatrix}$$

which is clearly impossible.

Let $\Gamma = \langle \sum a_i E_i, \sum b_i E_i, \sum c_i E_i \rangle$ be a three-dimensional subspace of $\mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1$. Suppose $E^2(\Gamma) \neq \{0\}$. In this case we may assume $a_2 = 1$ and $b_2 = c_2 = 0$. If $b_1 = c_1 = 0$, then

$$\psi = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -a_3 & 1 & 0 \\ 0 & -a_4 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma = \langle E_2, E_3, E_4 \rangle$. If $b_1 \neq 0$ or $c_1 \neq 0$, and $b_3 c_4 - b_4 c_3 \neq 0$, then we may assume $b_1 = 1$ and $c_1 = 0$ and hence

$$\psi = \begin{bmatrix} -b_4 c_3 + b_3 c_4 & a_1(b_4 c_3 - b_3 c_4) & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a_4 c_3 - a_3 c_4 & c_4 & -c_3 \\ 0 & a_3(-c_3 + c_4) - a_4(c_3 + c_4) & c_3 - c_4 & c_3 + c_4 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma = \langle E_1 + E_3, E_2, E_4 \rangle$. If $b_1 \neq 0$ or $c_1 \neq 0$, and $b_3 c_4 - b_4 c_3 = 0$, then we may again assume $b_1 = 1$ and $c_1 = 0$ and hence

$$\psi = \begin{bmatrix} 1 & -a_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a_4 c_3 - a_3 c_4 & c_4 & -c_3 \\ 0 & a_3(-c_3 + c_4) - a_4(c_3 + c_4) & c_3 - c_4 & c_3 + c_4 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma = \langle E_1, E_2, E_4 \rangle$. On the other hand, suppose $E^2(\Gamma) = \{0\}$. Then $\Gamma = \langle E_1, E_3, E_4 \rangle$. Hence we have that any three-dimensional subspace is equivalent to $\langle E_2, E_3, E_4 \rangle$, $\langle E_1 + E_3, E_2, E_4 \rangle$, $\langle E_1, E_2, E_4 \rangle$, or $\langle E_1, E_3, E_4 \rangle$. Once again, completeness and nonredundency can be verified as described in Appendix B.1.

B.3. Proof for Theorem 2.12 (algebra $\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$). We prove only the assertion that any proper subspace of $\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$ is equivalent to one of the subspaces listed. First, note that $\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$ admits exactly one family of invariant scalar products (ω_ρ); in coordinates

$$\omega_\rho = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \rho \end{bmatrix}, \quad \rho \neq 0. \tag{B.5}$$

Let $\Gamma = \langle \sum a_i E_i \rangle$ be a one-dimensional subspace. Suppose $E^4(\Gamma) = \{0\}$, i.e., $a_4 = 0$. The pseudo-orthogonal group $\mathsf{SO}\,(2,1)$ (as a subgroup of the group of automorphisms) acts transitively on the level sets $\mathcal{H}_\alpha = \{A \in \langle E_1, E_2, E_3 \rangle \,:\, \omega_0(A,A) = \alpha,\ A \neq 0\}$. ($\mathcal{H}_\alpha$ is a hyperboloid of two sheets when $\alpha < 0$, a hyperboloid of one sheet when $\alpha > 0$, and a punctured cone when $\alpha = 0$.) Hence, there exists an automorphism ψ such that $\psi \cdot (a_1 E_1 + a_2 E_2 + a_3 E_3)$ is either $\alpha E_1 \in \mathcal{H}_{\alpha^2}$, $\alpha E_3 \in \mathcal{H}_{-\alpha^2}$, or $E_1 + E_3 \in \mathcal{H}_0$. Consequently, Γ is equivalent to either $\langle E_1 \rangle$, $\langle E_3 \rangle$, or $\langle E_1 + E_3 \rangle$. Likewise, when $E^4(\Gamma) \neq \{0\}$, i.e., $a_4 \neq 0$, then Γ is equivalent to $\langle E_1 + E_4 \rangle$, $\langle E_3 + E_4 \rangle$, $\langle E_1 + E_3 + E_4 \rangle$, or $\langle E_4 \rangle$. Note that ω_0 is invariant under automorphisms (i.e., $\omega_0(\psi(A), \psi(B)) = \omega_0(A,B)$ for any automorphism ψ); also, $E^4(A) = 0$ if and only if $E^4(\psi \cdot A) = 0$. Accordingly, no two of the one-dimensional subspaces enumerated are equivalent.

Let $\Gamma = \langle A_1, A_2 \rangle$ be a two-dimensional subspace. The sign $\sigma(\Gamma)$ of Γ is given by

$$\sigma(\Gamma) = \operatorname{sgn}\left(\begin{vmatrix} \omega_0(A_1,A_1) & \omega_0(A_1,A_2) \\ \omega_0(A_1,A_2) & \omega_0(A_2,A_2) \end{vmatrix} \right).$$

It is easy to show that the sign of Γ does not depend on the parametrization of Γ and is invariant under automorphisms, i.e., $\sigma(\Gamma) = \sigma(\psi \cdot \Gamma)$ for any automorphism $\psi \in \mathsf{Aut}(\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1)$ (see [4]). Furthermore, the condition $E^4(\Gamma) = \{0\}$ is invariant under automorphisms. Suppose $E^4(\Gamma) \neq \{0\}$. Then $\Gamma \cap \langle E_1, E_2, E_3 \rangle$ is a one-dimensional subspace. Let $B \in \mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$ such that $\langle B \rangle = \Gamma \cap \langle E_1, E_2, E_3 \rangle$. Note that for any automorphism ψ we have that $(\psi \cdot \Gamma) \cap \langle E_1, E_2, E_3 \rangle = \psi \cdot (\Gamma \cap \langle E_1, E_2, E_3 \rangle) = \langle \psi \cdot B \rangle$. Hence, we have another sign for Γ, namely $\bar{\sigma}(\Gamma) = \operatorname{sgn}(\omega_0(B,B))$. (We have that $\bar{\sigma}(\Gamma)$ does not depend on the parametrization for Γ and is invariant under automorphisms.) We also note that the projection of Γ to $\mathfrak{g}_{3.6}$ is a two-dimensional subspace if and only if the same holds true for $\psi \cdot \Gamma$ for any automorphism ψ of $\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$.

If $E^4(\Gamma) = \{0\}$, then Γ is equivalent to $\langle E_1, E_3 \rangle$ whenever $\sigma(\Gamma) = -1$, Γ is equivalent to $\langle E_1, E_2 + E_3 \rangle$ whenever $\sigma(\Gamma) = 0$, and Γ is equivalent to $\langle E_1, E_2 \rangle$ whenever $\sigma(\Gamma) = 1$ (see [4]). Suppose $E^4(\Gamma) \neq \{0\}$. As $\mathsf{SO}\,(2,1)$ acts transitively on the level sets $\mathcal{H}_\alpha$, it follows that Γ is equivalent to $\langle E_3, a_1 E_1 + a_2 E_2 + E_4 \rangle$ (when $\bar{\sigma}(\Gamma) = -1$), $\langle E_2 + E_3, a_1 E_1 + a_2 E_2 + E_4 \rangle$ (when $\bar{\sigma}(\Gamma) = 0$), or $\langle E_1, a_2 E_2 + a_3 E_3 + E_4 \rangle$ (when $\bar{\sigma}(\Gamma) = 1$) for some $a_1, a_2, a_3 \in \mathbb{R}$.

Consider the subspace $\Gamma' = \langle E_3, a_1 E_1 + a_2 E_2 + E_4 \rangle$. If $a_1 = a_2 = 0$, then $\Gamma' = \langle E_3, E_4 \rangle$. On the other hand, if $a_1^2 + a_2^2 \neq 0$, then $\Gamma' =$

$\left\langle E_3, \cos\theta E_1 + \sin\theta E_2 + \frac{1}{r}E_4 \right\rangle$ for some $r, \theta \in \mathbb{R}$ and

$$\psi = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & r \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma' = \langle E_3, E_1 + E_4 \rangle$.

Next, consider the subspace $\Gamma' = \langle E_2 + E_3, a_1 E_1 + a_2 E_2 + E_4 \rangle$. If $a_1 = a_2 = 0$, then $\Gamma' = \langle E_2 + E_3, E_4 \rangle$. If $a_1^2 + a_2^2 \neq 0$ and $\sigma(\Gamma') = \operatorname{sgn}(-a_2^2) = 0$, then $a_2 = 0$ and so $\psi = \operatorname{diag}(1, 1, 1, a_1)$ is an automorphism such that $\psi \cdot \Gamma' = \langle E_2 + E_3, E_1 + E_4 \rangle$. If $a_1^2 + a_2^2 \neq 0$ and $\sigma(\Gamma') = -1$, then $a_2 \neq 0$ and

$$\psi = \begin{bmatrix} 1 & -\frac{a_1}{a_2} & \frac{a_1}{a_2} & 0 \\ \frac{a_1}{a_2} & 1 - \frac{a_1^2}{2a_2^2} & \frac{a_1^2}{2a_2^2} & 0 \\ \frac{a_1}{a_2} & -\frac{a_1^2}{2a_2^2} & 1 + \frac{a_1^2}{2a_2^2} & 0 \\ 0 & 0 & 0 & a_2 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma' = \langle E_2 + E_3, \frac{a_1^2}{2a_2}(E_2 + E_3) + a_2(E_2 + E_4) \rangle = \langle E_2 + E_3, E_2 + E_4 \rangle$. (Clearly the situation $\sigma(\Gamma) = 1$ is impossible.)

Lastly, consider that subspace $\Gamma' = \langle E_1, a_2 E_2 + a_3 E_3 + E_4 \rangle$. If $a_2 = a_3 = 0$, then $\Gamma = \langle E_1, E_4 \rangle$. If $a_2^2 + a_3^2 \neq 0$ and $\sigma(\Gamma') = \operatorname{sgn}(a_2^2 - a_3^2) = -1$, then $\Gamma' = \left\langle E_1, \sinh\theta E_2 + \cosh\theta E_3 + \frac{1}{r}E_4 \right\rangle$ for some $r, \theta \in \mathbb{R}$ and so

$$\psi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\theta & -\sinh\theta & 0 \\ 0 & -\sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 0 & r \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma' = \langle E_1, E_3 + E_4 \rangle$. If $a_2^2 + a_3^2 \neq 0$ and $\sigma(\Gamma') = \operatorname{sgn}(a_2^2 - a_3^2) = 1$, then $\Gamma' = \left\langle E_1, \cosh\theta E_2 + \sinh\theta E_3 + \frac{1}{r}E_4 \right\rangle$ for some $r, \theta \in \mathbb{R}$ and so

$$\psi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\theta & -\sinh\theta & 0 \\ 0 & -\sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 0 & r \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma' = \langle E_1, E_2 + E_4 \rangle$. If $a_2^2 + a_3^2 \neq 0$ and $\sigma(\Gamma') = \operatorname{sgn}(a_2^2 - a_3^2) = 0$, then $\Gamma' = \left\langle E_1, E_2 \pm E_3 + \frac{1}{r}E_4 \right\rangle$ for some $r \in \mathbb{R}$

and so $\psi = \mathrm{diag}(1,1,1,r)$ or $\psi = \mathrm{diag}(-1,1,-1,r)$ is an automorphisms such that $\psi \cdot \Gamma' = \langle E_1, E_2 + E_3 + E_4 \rangle$.

Hence, if $E^4(\Gamma) \neq \{0\}$, then Γ is equivalent to $\langle E_1, E_4 \rangle$, $\langle E_3, E_4 \rangle$, $\langle E_1, E_2 + E_4 \rangle$, $\langle E_1, E_3 + E_4 \rangle$, $\langle E_1 + E_4, E_3 \rangle$, $\langle E_2 + E_3, E_4 \rangle$, $\langle E_1, E_2 + E_3 + E_4 \rangle$, $\langle E_1 + E_4, E_2 + E_3 \rangle$, or $\langle E_2 + E_4, E_2 + E_3 \rangle$. The signs corresponding to these subspaces are given by

$\langle E_1 + E_4, E_3 \rangle$	$\sigma = -1$	$\bar{\sigma} = -1$
$\langle E_2 + E_4, E_2 + E_3 \rangle$	$\sigma = -1$	$\bar{\sigma} = 0$
$\langle E_1, E_3 + E_4 \rangle$	$\sigma = -1$	$\bar{\sigma} = 1$
$\langle E_3, E_4 \rangle$	$\sigma = 0$	$\bar{\sigma} = -1$
$\langle E_2 + E_3, E_4 \rangle$	$\sigma = 0$	$\bar{\sigma} = 0$
$\langle E_1 + E_4, E_2 + E_3 \rangle$	$\sigma = 0$	$\bar{\sigma} = 0$
$\langle E_1, E_4 \rangle$	$\sigma = 0$	$\bar{\sigma} = 1$
$\langle E_1, E_2 + E_3 + E_4 \rangle$	$\sigma = 0$	$\bar{\sigma} = 1$
$\langle E_1, E_2 + E_4 \rangle$	$\sigma = 1$	$\bar{\sigma} = 1.$

Subspaces corresponding to different signs clearly cannot be equivalent. The only pairs for which the signs match are $(\langle E_2 + E_3, E_4 \rangle, \langle E_1 + E_4, E_2 + E_3 \rangle)$ and $(\langle E_1, E_4 \rangle, \langle E_1, E_2 + E_3 + E_4 \rangle)$. In these cases nonequivalence follows from the invariant property of whether or not the projection of Γ to $\mathfrak{g}_{3.6}$ is a two-dimensional subspace.

Let Γ be a three-dimensional subspace. Again, the condition $E^4(\Gamma) = \{0\}$ is invariant under automorphisms. Suppose $E^4(\Gamma) \neq \{0\}$. Then $\Gamma \cap \langle E_1, E_2, E_3 \rangle$ is a two-dimensional subspace. Let $B, C \in \mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$ such that $\langle B, C \rangle = \Gamma \cap \langle E_1, E_2, E_3 \rangle$. For any automorphism ψ we have that $(\psi \cdot \Gamma) \cap \langle E_1, E_2, E_3 \rangle = \psi \cdot (\Gamma \cap \langle E_1, E_2, E_3 \rangle) = \langle \psi \cdot B, \psi \cdot C \rangle$. Hence, we shall define the sign $\bar{\bar{\sigma}}(\Gamma)$ of Γ as $\bar{\bar{\sigma}}(\Gamma) = \sigma(\langle B, C \rangle)$. (We have that $\bar{\bar{\sigma}}(\Gamma)$ does not depend on the parametrization for Γ and is invariant under automorphisms.) We also note that the projection of Γ to $\mathfrak{g}_{3.6}$ is $\mathfrak{g}_{3.6}$ if and only if the same holds true for $\psi \cdot \Gamma$ for any automorphism ψ of $\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$.

The orthogonal complement $\Gamma^{\perp}$ of Γ with respect to ω_1 (see (B.5)) is a one-dimensional subspace $\Gamma^{\perp} = \langle \sum a_i E_i \rangle$. By transitivity of $\mathsf{SO}\,(2,1)_0$ (the group of inner automorphisms) on each of the connected components of the level sets $\mathcal{H}_\alpha$, there exists an inner automorphism φ of $\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$ such that

$\varphi \cdot \Gamma$ is equal to one of the following subspaces

$$\langle E_1\rangle\,, \quad \langle E_3\rangle\,, \quad \langle E_1 + E_3\rangle\,, \quad \langle E_1 - E_3\rangle\,, \quad \langle E_4\rangle\,,$$
$$\langle E_1 + aE_4\rangle\,, \quad \langle E_3 + aE_4\rangle\,, \quad \langle E_1 + E_3 + aE_4\rangle\,, \quad \langle E_1 - E_3 + aE_4\rangle$$

for some $a \neq 0$. Hence, Γ is equivalent to the orthogonal complement of one of these one-dimensional subspaces with respect to ω_1; the respective orthogonal complements are

$$\langle E_2, E_3, E_4\rangle\,, \quad \langle E_1, E_2, E_4\rangle\,, \quad \langle E_1 + E_3, E_2, E_4\rangle\,, \quad \langle E_1 - E_3, E_2, E_4\rangle\,,$$
$$\langle E_1, E_2, E_3\rangle\,, \quad \left\langle E_2, E_3, E_1 - \tfrac{1}{a}E_4\right\rangle, \quad \left\langle E_1, E_2, E_3 + \tfrac{1}{a}E_4\right\rangle,$$
$$\left\langle E_1 + E_3, E_2, E_1 - \tfrac{1}{a}E_4\right\rangle, \quad \left\langle E_1 - E_3, E_2, E_1 - \tfrac{1}{a}E_4\right\rangle.$$

The automorphisms $\psi = \mathrm{diag}(1,1,1,\pm a)$ serve to normalize the E_4 components. Moreover, we have that $\psi = \mathrm{diag}(1,-1,-1,1)$ is an automorphism such that $\psi \cdot \langle E_1 - E_3, E_2, E_4\rangle = \langle E_1 + E_3, E_2, E_4\rangle$ and $\psi \cdot \langle E_1 - E_3, E_2, E_1 + E_4\rangle = \langle E_1 + E_3, E_2, E_1 + E_4\rangle$. Hence we have shown that any three-dimensional subspace is equivalent to one of the seven subspaces enumerated in the statement of the theorem. As $\langle E_1, E_2, E_3\rangle$ is the only subspace for which $E^4(\Gamma) = \{0\}$, it is not equivalent to any of the other six subspaces enumerated. For the remaining six subspaces we have that

$$\begin{aligned}\bar{\bar{\sigma}}(\langle E_2, E_3, E_4\rangle) = \bar{\bar{\sigma}}(\langle E_2, E_3, E_1 + E_4\rangle) = -1\\ \bar{\bar{\sigma}}(\langle E_1 + E_3, E_2, E_4\rangle) = \bar{\bar{\sigma}}(\langle E_1 + E_3, E_2, E_1 + E_4\rangle) = 0\\ \bar{\bar{\sigma}}(\langle E_1, E_2, E_4\rangle) = \bar{\bar{\sigma}}(\langle E_1, E_2, E_3 + E_4\rangle) = 1.\end{aligned}$$

Accordingly, by looking at the projection of each of the subspaces to $\mathfrak{g}_{3.6}$, we conclude that none of the subspaces are equivalent.

B.4. PROOF FOR THEOREM 2.13 (ALGEBRA $\mathfrak{g}_{3.7} \oplus \mathfrak{g}_1$). We prove only the assertion that any proper subspace is equivalent to exactly one of the subspaces listed. First note that $\mathfrak{g}_{3.7} \oplus \mathfrak{g}_1$ admits exactly one family of invariant scalar products (ω_ρ); in coordinates

$$\omega_\rho = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \rho \end{bmatrix}, \quad \rho \neq 0. \tag{B.6}$$

The orthogonal group $\mathsf{SO}\,(3)$ (i.e., the group of inner automorphisms) acts transitively on the spheres $\mathcal{S}_\alpha = \{A \in \langle E_1, E_2, E_3\rangle \,:\, \omega_0(A,A) = \alpha\}$, $\alpha > 0$. We note that for any subspace Γ of $\mathfrak{g}_{3.7} \oplus \mathfrak{g}_1$ we have that $E^4(\Gamma) = \{0\}$ if and only if $E^4(\psi \cdot \Gamma) = \{0\}$ for any automorphism ψ.

Let $\Gamma = \langle \sum a_i E_i \rangle$ be a one-dimensional subspace. If $E^4(\Gamma) = \{0\}$, then there exists an inner automorphism φ such that $\varphi \cdot \Gamma = \langle E_1 \rangle$. If $E^4(\Gamma) \neq \{0\}$ and $a_1^2 + a_2^2 + a_3^2 \neq 0$, then there exists an inner automorphism φ such that $\varphi \cdot \Gamma = \langle E_1 + aE_4\rangle$ for some $a \neq 0$. Furthermore, $\psi = \operatorname{diag}(1,1,1,\frac{1}{a})$ is an automorphism such that $\psi \cdot \langle E_1 + aE_4\rangle = \langle E_1 + E_4\rangle$. On the other hand, if $E^4(\Gamma) \neq \{0\}$ and $a_1^2 + a_2^2 + a_3^2 = 0$, then $\Gamma = \langle E_4 \rangle$.

Let Γ be a three-dimensional subspace. Its orthogonal complement $\Gamma^\perp$ with respect to ω_1 (see (B.6)) is a one-dimensional subspace; hence there exists an inner automorphism φ such that $\varphi \cdot \Gamma^\perp$ is $\langle E_1\rangle$, $\langle E_4 \rangle$, or $\langle E_1 + aE_4\rangle$. Thus Γ is equivalent to the orthogonal complements of one of these subspaces, namely, $\langle E_2, E_3, E_4\rangle$, $\langle E_1, E_2, E_3\rangle$, and $\left\langle E_1 - \frac{1}{a}E_4, E_2, E_3 \right\rangle$. For the last case we have that $\psi = \operatorname{diag}(1,1,1,a)$ is an automorphism such that $\psi \cdot \left\langle E_1 - \frac{1}{a}E_4, E_2, E_3\right\rangle = \langle E_1 - E_4, E_2, E_3\rangle$.

Let $\Gamma = \langle \sum a_i E_i, \sum b_i E_i \rangle$ be a two-dimensional subspace of $\mathfrak{g}_{3.7} \oplus \mathfrak{g}_1$. If $E^4(\Gamma) = \{0\}$, then Γ is a subspace of $\langle E_1, E_2, E_3\rangle$ and so there exists an inner automorphism φ such that $\varphi \cdot \Gamma = \langle E_1, E_2 \rangle$ ([4]). Suppose $E^4(\Gamma) \neq \{0\}$. We may assume $a_4 = 0$ and $b_4 \neq 0$. Hence there exists an inner automorphism φ such that $\varphi \cdot \Gamma = \langle E_1, \sum b_i' E_i\rangle$ with $b_1' = 0$ and $b_4' \neq 0$. If $b_2' = b_3' = 0$, then $\varphi \cdot \Gamma = \langle E_1, E_4 \rangle$. On the other hand if $b_2'^2 + b_3'^2 \neq 0$, then $\varphi \cdot \Gamma = \langle E_1, \cos\theta E_2 + \sin\theta E_3 + rE_4\rangle$ for some $\theta, r \in \mathbb{R}$ and so

$$\psi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & \frac{1}{r} \end{bmatrix}$$

is an automorphism such that $\psi \cdot \varphi \cdot \Gamma = \langle E_1, E_2 + E_4\rangle$.

It is a simple matter to verify that no two of the subspaces enumerated are equivalent.

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References

[1] R.M. Adams, R. Biggs, C.C. Remsing , Control systems on the orthogonal group $\mathsf{SO}(4)$, *Commun. Math.* **21** (2013), 107 – 128.

[2] D.I. Barrett, R. Biggs, C.C. Remsing, Affine distributions on a four-dimensional central extension of the semi-Euclidean group, to appear i Note Mat. 35 (2015).

[3] R. Biggs, P. Nagy, On extensions of sub-Riemannian structures on Lie groups (preprint).

[4] R. Biggs, C.C. Remsing, Control affine systems on semisimple three-dimensional Lie groups, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)* **59** (2013), 399 – 414.

[5] R. Biggs, C.C. Remsing, Control affine systems on solvable three-dimensional Lie groups, I, *Arch. Math. (Brno)* **49** (2013), 187 – 197.

[6] R. Biggs, C.C. Remsing, Control affine systems on solvable three-dimensional Lie groups, II, *Note Mat.* **33** (2013), 19 – 31.

[7] R. Biggs, C.C. Remsing, Control systems on three-dimensional Lie groups: equivalence and controllability, *J. Dyn. Control Syst.* **20** (2014), 307 – 339.

[8] R. Biggs, C.C. Remsing, Cost-extended control systems on Lie groups, *Mediterr. J. Math.* **11** (2014), 193 – 215.

[9] R. Biggs, C.C. Remsing, Some remarks on the oscillator group, *Differential Geom. Appl.* **35** (2014), 199 – 209.

[10] T. Christodoulakis, G.O. Papadopoulos, A. Dimakis, Automorphisms of real four-dimensional Lie algebras and the invariant characterization of homogeneous 4-spaces, *J. Phys. A* **36** (2003), 427 – 441.

[11] R. Ghanam, I. Strugar, G. Thompson, Matrix representations for low dimensional Lie algebras, *Extracta Math.* **20** (2005), 151 – 184.

[12] N. Iizuka, S. Kachru, N. Kundu, P. Narayan, N. Sircar, S.P. Trivedi, H. Wang, Extremal horizons with reduced symmetry: hyperscaling violation, stripes, and a classification for the homogeneous case, *Journal of High Energy Physics* **2013** (2013), 126.

[13] A. Krasiński, C.G. Behr, E. Schücking, F.B. Estabrook, H.D. Wahlquist, G.F.R. Ellis, R. Jantzen, W. Kundt, The Bianchi classification in the Schücking-Behr approach, *Gen. Relativity Gravitation* **35** (2003), 475 – 489.

[14] M.A.H. MacCallum, On the classification of the real four-dimensional Lie algebras, in "On Einstein's Path" (New York University, 1996), Springer, New York, 1999, 299 – 317.

[15] F.M. Mahomed, P.G.L. Leach, Lie algebras associated with scalar second-order ordinary differential equations, *J. Math. Phys.* **30** (12) (1989), 2770 – 2777.

[16] G.M. Mubarakzjanov, On solvable Lie algebras, *Izv. Vysš. Učehn. Zaved. Matematika* **1963** no 1 (32) (1963), 114 – 123.

[17] J. Patera, R.T. Sharp, P. Winternitz, H. Zassenhaus, Invariants

of real low dimension Lie algebras, *J. Mathematical Phys.* **17** (6) (1976), 986 – 994.

[18] J. Patera, P. Winternitz, Subalgebras of real three- and four-dimensional Lie algebras, *J. Mathematical Phys.* **18** (7) (1977), 1449 – 1455.

[19] R.O. Popovych, V.M. Boyko, M.O. Nesterenko, M.W. Lutfullin, Realizations of real low-dimensional Lie algebras, *J. Phys. A* **36** (26) (2003), 7337 – 7360.

[20] R.O. Popovych, N.M. Ivanova, New results on group classification of nonlinear diffusion-convection equations, *J. Phys. A* **37** (30) (2004), 7547 – 7565.

[21] O.O. Vaneeva, A.G. Johnpillai, R.O. Popovych, C. Sophocleous, Enhanced group analysis and conservation laws of variable coefficient reaction-diffusion equations with power nonlinearities, *J. Math. Anal. Appl.* **330** (2007), 1363 – 1386.