EXTRACTA MATHEMATICAE
Vol. **34**, Num. 1 (2019), 99 – 122

doi:10.17398/2605-5686.34.1.99
Available online March 13, 2019

Tetrahedral chains and a curious semigroup

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Received January 7, 2019
Accepted February 27, 2019

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Abstract: In 1957 Steinhaus asked for a proof that a chain of identical regular tetrahedra joined face to face cannot be closed. Świerczkowski gave a proof in 1959. Several other proofs are known, based on showing that the four reflections in planes though the origin parallel to the faces of the tetrahedron generate a group $\mathcal{R}$ isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$. We relate the reflections to elements of a semigroup of 3×3 matrices over the finite field $\mathbb{Z}_3$, whose structure provides a simple and transparent new proof that $\mathcal{R}$ is a free product. We deduce the non-existence of a closed tetrahedral chain, prove that $\mathcal{R}$ is dense in the orthogonal group $\mathbb{O}(3)$, and show that every $\mathcal{R}$-orbit on the 2-sphere is equidistributed.

Key words: tetrahedral chain, free product, semigroup, density, equidistribution, spherical harmonic, Cayley graph.

AMS *Subject Class.* (2010): 20F55, 20M05, 22E46, 37A30, 42C10, 43A15.

1. Introduction

In 1957 Hugo Steinhaus contributed two related questions to the Problems section of *Colloquium Mathematicum* [21]. In loose translation from the French, they were:

> **P 175**. The image of a regular tetrahedron T_1 (fixed in Euclidean space of 3 dimensions) under reflection in one of its faces gives the tetrahedron T_2. Iteration gives rise to a sequence of pairwise congruent tetrahedra $\{T_n\}$. Supposing that each face serves as a mirror only once, demonstrate that:
>
> (1) $m \neq n$ implies $T_m \neq T_n$,
> (2) Whatever the region R may be, there exists a sequence of tetrahedra $\{T_n\}$ such that the set of vertices is dense in R.

Steinhaus indicated that this problem is from the *New Scottish Book*, Problem 290 1.III.1956. The original *Scottish Book* was a notebook of open

ISSN: 0213-8743 (print), 2605-5686 (online)

mathematical problems compiled by regular visitors to the famous Scottish Café in what is now Lviv, Ukraine. An English typescript is available [28], and the original can be viewed online as Volume 0 at [30]. When World War II ended, Steinhaus revived the book as the *New Scottish Book*, Volumes 1 and 2 at [30].

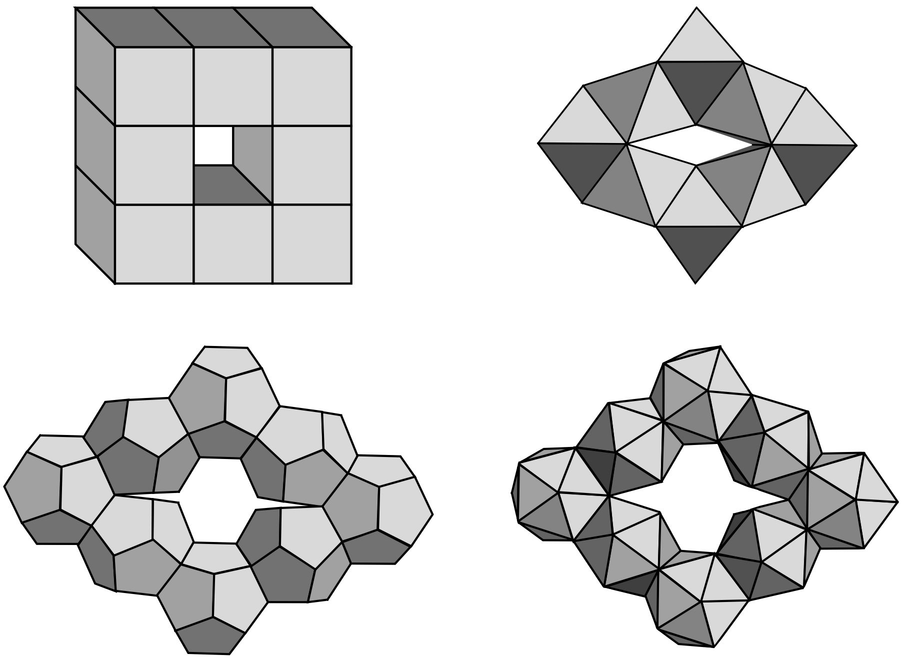

Figure 1: Closed chains of cubes, octahedra, dodecahedra, and icosahedra. Adapted from [10].

As far as we are aware, statement (2) is still open. Its analogue for cubes is clearly false, but its analogue for the other regular solids is a plausible conjecture. In particular, their dihedral angles are not rational multiples of π.

Statement (1) implies that a chain of identical regular tetrahedra, joined face to face, cannot be closed. In contrast, it is easy to find closed chains for the other four regular polyhedra, Figure 1. (One key difference is that unlike the tetrahedron, opposite faces of these polyhedra are parallel.) Stanisław Świerczkowski subsequently proved that no closed chain of tetrahedra exists, aside from trivial examples where consecutive tetrahedra coincide. In [24] he proved that two particular rotations in $\mathbb{R}^3$ generate a free group on two generators, and stated as a corollary that this result disproves the existence of a closed chain of regular tetrahedra. He wrote: "This corollary gives a positive answer to a question of H. STEINHAUS .... However we shall not prove it here." In [25] he completed the proof by explaining the connection with chains of tetrahedra.

The rotations concerned are

$$\begin{bmatrix} \frac{1}{3} & -\frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}.$$

Their axes are at right angles to each other and both are rotations through $\cos^{-1}(\frac{1}{3})$. Świerczkowski's proof that these matrices generate a free group uses induction on a sequence of integers determined by the two matrices, and his main aim is to prove that these are not divisible by 3. In passing, we mention that this group-theoretic result can also be used as the basis of a proof of the famous Banach-Tarski paradox [27]: a solid ball in $\mathbb{R}^3$ can be dissected into finitely many disjoint subsets, which can be fitted together via rigid motions to create two solid balls, each congruent to the original one. The free product group discussed below can also be used in this manner.

Dekker [8] and Mason [19] sketched new proofs that no closed tetrahedral chain exists, based on the idea that the group generated by reflections in the four planes through the origin parallel to the faces of the tetrahedron is isomorphic to a free product $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$. (Without loss of generality, the barycentre of the first tetrahedron in the chain is the origin. It is important to distinguish these linear reflections from the affine reflections in the faces of the tetrahedron, which do not fix the origin; see Subsection 2.2.) Tomkowicz and Wagon [27, Theorem 3.10] represents the four reflections as 4×4 matrices using barycentric coordinates, and analyse an arbitrarily long product of these matrices. The entries of such a product are polynomials, evaluated at $\pm\frac{1}{3}$, $\pm\frac{2}{3}$. As in [24], the key step in an inductive proof again involves divisibility by a power of 3.

Say that a chain of tetrahedra is *embedded* if distinct tetrahedra are disjoint except for the common face of consecutive members of the chain. All of the above proofs rule out the existence of nontrivial closed chains, embedded or not. These proofs are relatively short and simple, but none is particularly transparent. In Section 3 we present a new proof, with a clear storyline that emphasises the role of the integer 3. We use Cartesian coordinates, but it is possible to recast the discussion using the more traditional barycentric coordinates. With a convenient choice of the initial tetrahedron in $\mathbb{R}^3$, with vertices at four corners of the cube $[-1,1]^3$, the 3×3 matrices representing the four reflections have rational entries with denominator 1 or 3; see Section 2. That they generate $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ is a reformulation of the statement that the product of any nontrivial sequence of the four matrices (that is, avoiding consecutive repetitions of the same matrix), other than the empty sequence, can never be the identity.

For a contradiction, suppose such a sequence exists. Let the group generated by the four reflections be $\mathcal{R}$, which is a subgroup of the orthogonal group $\mathbb{O}(3)$ acting on $\mathbb{R}^3$. If each matrix is multiplied by 3 it has integer entries, and the corresponding product must be the identity multiplied by 3^k where $k \geq 1$ is the length of the sequence. These products no longer form a group, but together with the zero matrix they form a semigroup. Reducing modulo 3, a nontrivial product of the corresponding matrices over $\mathbb{Z}_3$ must be the zero matrix. Theorem 3.2 proves that the four reflection matrices (mod 3) generate an order-33 semigroup. This contains the zero matrix $\mathbf{0}$, but we prove that no nontrivial product of its nonzero elements is zero. (As before, 'nontrivial' means no generator appears twice consecutively.) Indeed, every matrix in the semigroup other than $\mathbf{0}$ has *all* entries equal to ± 1 (mod 3). This contradiction proves that $\mathcal{R}$ is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$. (An isomorphic semigroup can be obtained using barycentric coordinates, and the proof can be also be expressed in that framework.)

The non-existence of a nontrivial closed chain of regular tetrahedra follows easily, using essentially the argument of Świerczkowski [25], which constructs the chain using successive reflections in faces. The translations in the Euclidean group $\mathbb{E}(3)$ form a normal subgroup and can be factored out, reducing the geometric features required here to sequences of reflections. The four reflections lie in the quotient group $\mathbb{O}(3)$. A sequence of reflections determines a unique chain of face-to-face tetrahedra, and a nontrivial sequence determines a nontrivial chain. If this chain closes up, the corresponding sequence fixes the initial tetrahedron. There is one subtlety, discussed briefly in Subsection 3.1: this sequence can fix the tetrahedron setwise rather than pointwise. That is, it belongs to the symmetry group of the tetrahedron, but need not be the identity. There are two ways to deal with this possibility. One is to observe that some power of the sequence must then be the identity (it is also necessary to deal with possible repetitions if the sequence starts and ends with the same reflection: this leads to a shorter chain, but after a series of such cancellations it turns out that the result must be nontrivial if the original chain is). The other, employed here, is to check that the semigroup proof remains valid if we replace the identity by a symmetry of the tetrahedron, because all such symmetries correspond to integer matrices. After multiplying by 3 and reducing modulo 3, these all become the zero matrix and the same proof works. Section 4 adds extra information about this semigroup.

In the absence of a closed chain, a natural question, also asked by Świerczkowski [26], arises: can *almost* closed chains be formed, in the sense that the gap between the initial and final tetrahedra can be made as small as we please? This question is connected to Steinhaus's statement (2), but it involves the faces of the tetrahedra, not just individual vertices. Elgersma and Wagon [9] give an affirmative answer for non-embedded chains, based on Kronecker's Theorem [1, 17] that if θ is an irrational multiple of π, the set $\{e^{ni\theta} : n \in \mathbb{Z}\}$ is dense in the unit circle $\mathbb{S}^1 \subseteq \mathbb{C}$. The most interesting case arises when the chain is embedded. Elgersma and Wagon [10, 11] prove the existence of closed embedded chains with arbitrarily small gaps. Their construction begins with a Boerdijk-Coxeter helix [4, 7], also named the tetrahelix by Fuller [12]. This is a linear chain of identical regular tetrahedra, all of whose vertices lie on a cylinder. This chain is generated by periodically repeating reflections in four distinct faces. They then construct a 'quadrahelix' by joining four copies of a tetrahelix of length $L+1$, overlapping them at a common end tetrahedron at the first and third joins, and attaching them face to face at the second join, so that the overall chain has reflectional symmetry about its midpoint. They prove that if $L = q - 1$ where p/q is a convergent of the continued fraction of $\theta = \cos^{-1}(\frac{2}{3})/(2\pi)$, the quadrahelix has the approximate form of a rhombus, and is almost closed, with the size of the gap tending to zero as q increases. For example when $L = 601{,}944$ the gap has size 1.3×10^{-7}.

Here we prove two related results, which do not prove the existence of almost closed chains but have independent interest. Section 5 gives a simple proof that $\mathcal{R}$ is dense in $\mathbb{O}(3)$. Steinhaus's statement (2) asks for more: the group generated by the *affine* reflections in the faces of a fixed regular tetrahedron has a dense orbit in $\mathbb{R}^3$. Taking account of the translations is more difficult, in part because the Euclidean group $\mathbb{E}(3)$ in R^3 is non-compact. Our density result is too weak to prove statement (2), and has no obvious consequences for almost-closed chains of tetrahedra, because it factors out translations.

Finally, in Section 6, we use the $\mathbb{O}(3)$ density result to prove a stronger theorem: the $\mathcal{R}$-orbit of any point of the unit 2-sphere $\mathbb{S}^2$ is equidistributed with respect to normalised Lebesgue surface measure on $\mathbb{S}^2$, where the density of $\mathcal{R}$ is defined using the limit of the proportion of words in the generating reflections that lie in a given open subset of $\mathbb{S}^2$, as the length of the words tends to infinity. The proof is an adaptation of a method of Arnold and Krylov [2], and can be viewed in the context of ergodic theory of non-abelian group actions; see Gorodnik and Nevo [13].

2. Reflections in faces of the tetrahedron

Elgersma and Wagon [9, 10, 11] pose the problem using barycentric coordinates and 4×4 matrices. Babiker and Janeczko [3] analyse chains of regular tetrahedra using a tensor product representation, combining the translations and reflections, to prove several new results. However, the reflections are again represented using barycentric coordinates. In this paper we use Cartesian coordinates and 3×3 matrices.

We choose a fixed reference tetrahedron $\Delta \subseteq \mathbb{R}^3$ with vertices

$$\begin{aligned} p_0 &: (1,1,1)\,, \\ p_1 &: (1,-1,-1)\,, \\ p_2 &: (-1,1,-1)\,, \\ p_3 &: (-1,-1,1)\,. \end{aligned}$$

The barycentre is at the origin. See Figure 2.

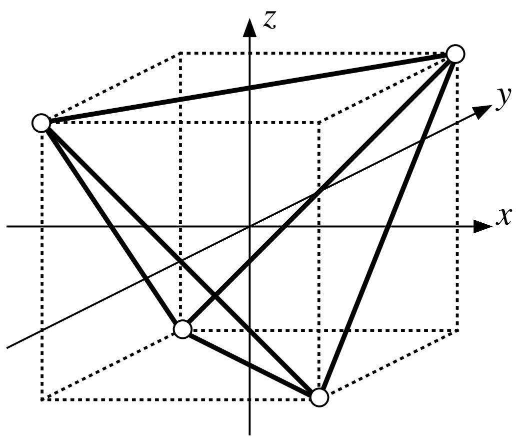

Figure 2: The cube $[-1,1]^3$ and the reference tetrahedron Δ.

2.1. Symmetries of the tetrahedron. The eight points $(\pm 1, \pm 1, \pm 1)$ are the vertices of a cube. The symmetry group of the cube consists of all permutations of the coordinates (x, y, z) together with sign changes on any coordinate, so has order 48. The symmetry group $\mathrm{Sym}(\Delta)$ of the tetrahedron is the subgroup in which the number of minus signs is even, and has order 24. It is, of course, isomorphic to the symmetric group $\mathbb{S}_4$, which permutes the four vertices.

Lemma 2.1. *All matrices in the symmetry group* $\mathrm{Sym}(\Delta)$ *have integer entries.*

Proof. All 3×3 matrices in the symmetry group of the cube, hence also of the tetrahedron, are signed permutation matrices, with entries $0, \pm 1$. ∎

2.2. The 3×3 matrices. Next, we compute the four reflection matrices and associated translations. The faces $F[p_i p_j p_k]$ through vertices p_i, p_j, p_k have equations:

$$\begin{aligned} F[p_1p_2p_3] &: \ X + Y + Z = -1\,, \\ F[p_0p_1p_2] &: \ X + Y - Z = 1\,, \\ F[p_0p_1p_3] &: \ X - Y + Z = 1\,, \\ F[p_0p_2p_3] &: \ -X + Y + Z = 1\,. \end{aligned}$$

Midpoints of centres of these faces are

$$\begin{aligned} q_0 &: \ \left(-\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}\right), \\ q_1 &: \ \left(\tfrac{1}{3}, \tfrac{1}{3}, -\tfrac{1}{3}\right), \\ q_2 &: \ \left(\tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{1}{3}\right), \\ q_3 &: \ \left(-\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right). \end{aligned}$$

Let M_i be reflection in face i, where $0 \le i \le 3$. Then it is the identity on that face, and reverses the line perpendicular to the face at its centre. This line joins the midpoint of the face to the remaining vertex.

Begin with the plane $P = F[p_1p_2p_3]$. The perpendicular is $(a, a, a) : a \in \mathbb{R}$. A general point

$$Y = (U, V, W) = (a, a, a) + Z$$

where $Z = (U - a, V - a, W - a) \in P$. Therefore

$$U + V + W - 3a = -1$$

so

$$a = \frac{U+V+W+1}{3}\,.$$

Then, solving for Z,

$$Z = \begin{bmatrix} \frac{2}{3}U - \frac{1}{3}V - \frac{1}{3}W - \frac{1}{3} \\ \frac{2}{3}V - \frac{1}{3}U - \frac{1}{3}W - \frac{1}{3} \\ \frac{2}{3}W - \frac{1}{3}U - \frac{1}{3}V - \frac{1}{3} \end{bmatrix}.$$

This maps via M_0 to $Z - (a, a, a)$, which is

$$M_0 Y = \begin{bmatrix} \frac{1}{3}U - \frac{2}{3}V - \frac{2}{3}W - \frac{2}{3} \\ \frac{1}{3}V - \frac{2}{3}U - \frac{2}{3}W - \frac{2}{3} \\ \frac{1}{3}W - \frac{2}{3}U - \frac{2}{3}V - \frac{2}{3} \end{bmatrix} = R_0 Y + T_0\,,$$

where

$$R_0 = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}, \qquad T_0 = \begin{bmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}.$$

Repeating similar calculations for the other three faces, we obtain:
For face $F[p_0p_1p_2]$:

$$R_1 = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}, \qquad T_1 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}.$$

For face $F[p_0p_1p_3]$:

$$R_2 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}, \qquad T_2 = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}.$$

For face $F[p_0p_2p_3]$:

$$R_3 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}, \qquad T_3 = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}.$$

The R_i are orthogonal, and T_i points perpendicular to the face concerned. Since $\mathrm{Sym}(\Delta)$ permutes the faces of Δ and fixes the origin, the R_i are conjugate under symmetries of Δ:

$$\sigma R_i \sigma^{-1} = R_{\sigma(i)}\,, \qquad \sigma \in \mathrm{Sym}(\Delta) \equiv \mathbb{S}_4\,.$$

2.3. Factoring out translations. The construction of chains of tetrahedra is intimately related to the subgroup of $\mathbb{O}(3)$ generated by the four reflection matrices, and for many purposes we can ignore the translations.

Definition 2.2. The group $\mathcal{R}$ is the subgroup of $\mathbb{O}(3)$ generated by the R_i for $0 \le j \le 3$.

The corresponding affine maps are

$$M_i Y = R_i Y + T_i \,, \qquad 0 \le i \le 3 \,.$$

The group $\mathbb{E}(3)$ of all rigid motions of $\mathbb{R}^3$ is a semidirect product $\mathbb{E}(3) = \mathbb{R}^3 \ltimes \mathbb{O}(3)$ where $\mathbb{R}^3$ is the normal subgroup of translations. The homomorphism $\mathbb{E}(3) \to \mathbb{O}(3)$ that factors out $\mathbb{R}^3$ maps (T_i, R_i) to R_i. Therefore a product of affine maps

$$M_{i_k} M_{i_{k-1}} \cdots M_{i_2} M_{i_1}$$

is a map of the form

$$R_{i_k} R_{i_{k-1}} \cdots R_{i_2} R_{i_1} + W_{i_k i_{k-1} \ldots i_2 i_1} \tag{2.1}$$

where $W_{i_k i_{k-1} \ldots i_2 i_1}$ is a translation determined by the semidirect product structure, which we do not state explicitly.

The product of the reflections R_i gives the orientation of the image tetrahedron; the translation $W_{i_k i_{k-1} \ldots i_2 i_1}$ leaves the orientation invariant. The sequence of translations is implicit in the sequence of reflections because exactly one pair of faces matches at each stage.

3. Free product structure

We now give a simple, structural proof of the main result of Dekker [8] and Mason [19]:

Theorem 3.1. *The group $\mathcal{R}$ is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$.*

Our proof is based on a rather curious semigroup, and we discuss this first. To get rid of fractions, define $Q_i = 3R_i$. We can then reduce modulo 3, to get matrices

$$S_i = Q_i \pmod 3 \,, \qquad 0 \le i \le 3 \,,$$

which are:

$$S_0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad S_1 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \qquad S_3 = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

Here the ± 1 lie in $\mathbb{Z}_3$, but in fact the calculations reported below also apply in $\mathbb{Z}$, except when the zero matrix arises and some entries may be multiples of 3.

The twelve products S_iS_j $(i \neq j)$ are:

$$S_0S_1 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}, \qquad S_0S_2 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix},$$

$$S_0S_3 = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}, \qquad S_1S_0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix},$$

$$S_1S_2 = \begin{bmatrix} -1 & 1 & -1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \qquad S_1S_3 = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix},$$

$$S_2S_0 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad S_2S_1 = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix},$$

$$S_2S_3 = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}, \qquad S_3S_0 = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$S_3S_1 = \begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}, \qquad S_3S_2 = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}.$$

Clearly $Q_i^2 = 9R_i^2 = 9I$, so $S_i^2 = \mathbf{0}$. Observe that the sixteen matrices S_i and S_iS_j $(i \neq j)$ are distinct, and distinct from their negatives $-S_i$ and $-S_iS_j$ $(i \neq j)$. Let $\mathcal{S}$ be the set of these 32 matrices. Case-by-case analysis shows that the S_i satisfy the following relations:

$$\begin{aligned} S_iS_jS_i &= S_i \qquad (i \neq j)\,, \\ S_iS_jS_k &= -S_iS_k \qquad (i \neq j,\ i \neq k,\ j \neq k)\,. \end{aligned} \tag{3.1}$$

(Using $\mathrm{Sym}(\Delta)$ we can reduce this calculation to the special case $i = 0$, $j = 1$, $k = 2$.)

Let $\mathcal{S}$ be the set of all of the above 32 matrices together with the zero matrix, so that

$$\mathcal{S} = \big\{\mathbf{0},\ \pm S_i,\ \pm S_iS_j\ :\ 0 \leq i,\ j \leq 3,\ i \neq j\big\}.$$

Theorem 3.2. (a) *The set $\mathcal{S}$ is a semigroup.*

(b) *The product of two nonzero members of $\mathcal{S}$ is nonzero, except for the trivial cases*

$$S_iS_i = \mathbf{0}\,, \qquad S_i(S_iS_j) = \mathbf{0}\,, \qquad (S_iS_j)S_j = \mathbf{0}\,, \qquad (S_iS_j)(S_jS_k) = \mathbf{0}\,,$$

and similar products involving minus signs.

Proof. For (a) we must show that all products of nonzero elements of $\mathcal{S}$ lie in $\mathcal{S}$. For (b) we must also show these products are nonzero. Both follow from a case-by-case check.

For products S_iS_j this is clear. Products of the form $S_i(S_jS_k)$ and $(S_iS_j)S_k$ are taken care of directly by the relations (3.1). Those relations also imply that when $i \neq j$, $k \neq l$ we have

$$(S_iS_j).(S_kS_l) \;=\; \begin{cases} \mathbf{0} & \text{if } j = k\,, \\ S_i & \text{if } j \neq k\,,\ k = i\,, \\ -S_iS_l & \text{if } j \neq k\,,\ k \neq i\,. \end{cases}$$

When $j = k$ the string $S_iS_jS_kS_l = S_iS_jS_jS_l$ is trivial. ∎

Note in particular that when $i \neq j$

$$(S_iS_j)^2 = (S_iS_jS_i)S_j = S_iS_j$$

which is nonzero, unlike squares of the S_i.

We are now ready to give the:

Proof of Theorem 3.1. The four free factors $\mathbb{Z}_2$ are generated respectively by R_0, R_1, R_2, R_3. We claim that the only relations between these generators are $R_i^2 = I$, where I is the identity.

Using the relations $R_i^2 = I$ we can write any element $\gamma \in \mathcal{R}$ in the form

$$\gamma = R_{i_k} R_{i_{k-1}} \cdots R_{i_2} R_{i_1}$$

where $R_{i_j} \neq R_{i_{j+1}}$ for all $1 \leq j \leq k-1$. We claim this representation as a word is unique.

If not, some nontrivial word is equal to the identity I. The corresponding nontrivial word in the Q_i of length l is equal to $3^l I$. Modulo 3, this word becomes zero. Consider the corresponding word in the S_i, which is also nontrivial:

$$w = S_{i_k} S_{i_{k-1}} \cdots S_{i_2} S_{i_1}\,.$$

By Theorem 3.2, w lies in $\mathcal{S} \setminus \{\mathbf{0}\}$, so all of its entries are $\pm 1 \pmod 3$. Therefore $w \neq \mathbf{0}$, so no nontrivial word in the R_i can be the identity. ∎

3.1. Non-existence of a closed chain. It is well known that Theorem 3.1 implies the non-existence of a nontrivial closed chain of regular tetrahedra. For completeness, we give a proof.

Theorem 3.3. *No nontrivial closed chain of tetrahedra exists.*

Proof. Suppose, for a contradiction, that there is such a chain. Consider the corresponding product of reflections R_i in $\mathcal{R}$.

Because each reflection R_i fixes the origin, the construction of the chain of tetrahedra corresponding to a given element

$$R_{j_k} R_{j_{k-1}} \cdots R_{j_2} R_{j_1} \in \mathcal{R}$$

does not add successive tetrahedra to an otherwise stationary chain. Instead, the chain corresponding to $R_{j_{k-1}} \cdots R_{j_2} R_{j_1}$ is reflected by R_{j_k}, and then translated by an appropriate amount so that it joins to the corresponding face of the reference tetrahedron Δ. Thus the chain at stage k has the structure

$$\begin{aligned}\Delta \to R_{j_k}\Delta \to R_{j_k}R_{j_{k-1}}\Delta \to R_{j_k}R_{j_{k-1}}R_{j_{k-2}}\Delta \to \cdots \\ \cdots \to R_{j_k}R_{j_{k-1}} \cdots R_{j_2}R_{j_1}\Delta\end{aligned}$$

where the arrow indicates ‘joins at a face’. (An alternative approach, in which each new face is added to a growing but otherwise static chain, is geometrically more natural but involves conjugates of the R_i, so this convention is a little simpler algebraically.)

A necessary condition for the chain to close up is then that there some nontrivial product of reflections is a symmetry of the tetrahedron:

$$R_{j_k} R_{j_{k-1}} \cdots R_{j_2} R_{j_1} = A$$

where $A \in \mathrm{Sym}(\Delta)$. Then the corresponding nontrivial product in $\mathcal{S}$ satisfies

$$S_{j_k} S_{j_{k-1}} \cdots S_{j_2} S_{j_1} = 3^{k+1} A = \mathbf{0}$$

by Lemma 2.1. This yields the same contradiction as in the proof of Theorem 3.1. ∎

4. Properties of the semigroup

The semigroup $\mathcal{S}$ has a lot of structure, which the calculations do not explain. We briefly investigate some of its features. The results of this section are not used later, but they help to explain some aspects of the structure of $\mathcal{S}$ from a different point of view.

The $\pm S_i$ are symmetric matrices, whereas the $\pm S_i S_j$ are not symmetric. There are $2^9 = 512$ matrices of size 3×3 with entries ± 1. The 32 such matrices in $\mathcal{S} \setminus \{\mathbf{0}\}$ are distinguished by the following properties:

(1) All entries are ± 1.

(2) The matrix has a repeated row, and the remaining row is either the same as the repeated row or the negative of the repeated row.

(3) The same goes for columns.

It is easy to prove that for matrices satisfying (1), condition (2) holds if and only if (3) does. We omit the proof.

Proposition 4.1. *The equivalent conditions* (1)+(2) *or* (1)+(3) *characterise the* 32 *nonzero elements of the semigroup* $\mathcal{S}$.

Proof. This follows from the list of elements, but we now give an independent proof avoiding case-by-case checking. We count now many such matrices exist. Observe that there are 8 possibilities for the first row R_1. The second and third rows R_2, R_3 are all possible choices of $R_2 = \pm R_1$, $R_3 = \pm R_1$, with four choices of the $\pm$ signs, so in total there are $8 \times 4 = 32$ such matrices. In other words, the equivalent conditions (1)+(2) or (1)+(3) characterise the elements of $\mathcal{S}$ that are not $\mathbf{0}$. ∎

Proposition 4.1 lets us give an alternative proof that $\mathcal{S}$ is a semigroup, without listing all products:

Theorem 4.2. *The collection of matrices satisfying* (1)+(2), *together with the zero matrix, is a semigroup.*

Proof. As observed, conditions (1) and (2) also imply (3). Redefine $\mathcal{S}$ to be the set of matrices satisfying these conditions, together with $\mathbf{0}$. Let $A, B \in \mathcal{S}$. Clearly $\mathbf{0} = \mathbf{0}.\mathbf{0} = \mathbf{0}.A = A.\mathbf{0}$. It remains to show that $AB \in \mathcal{S}$ when $A, B \neq \mathbf{0}$. Permuting rows we can write

$$A = P \begin{bmatrix} X \\ X \\ \varepsilon X \end{bmatrix}, \qquad \varepsilon = \pm 1\,,$$

where P is a permutation matrix, and $X = [x, y, z]$ is a row vector. Dually, permuting columns we can write

$$B = \begin{bmatrix} Y & Y & \delta Y \end{bmatrix} Q\,, \qquad \delta = \pm 1\,,$$

where Q is a permutation matrix, and $Y = [u, v, w]^{\mathrm{T}}$ is a column vector. Then

$$AB = P \begin{bmatrix} X \cdot Y & X \cdot Y & \delta X \cdot Y \\ X \cdot Y & X \cdot Y & \delta X \cdot Y \\ \varepsilon X \cdot Y & \varepsilon X \cdot Y & \varepsilon\delta X \cdot Y \end{bmatrix} Q\,.$$

Either $X \cdot Y = \mathbf{0}$ and AB is the zero matrix, or $X \cdot Y = \pm 1$ and the matrix in the middle clearly also satisfies (1) and (2). Now P and Q permute its rows and columns, leaving properties (1) and (2) unchanged. ∎

Remark 4.3. As stated above, the four matrices S_i are distinguished from the twelve matrices $S_i S_j$ $(i \neq j)$ by symmetry. For the symmetric matrices S_i, we have $X = Y$ and $X.Y = X.X = (\pm 1)^2 + (\pm 1)^2 + (\pm 1)^2 \equiv 0 \pmod 3$. For the asymmetric matrices $S_i S_j$ $(i \neq j)$ this does not happen, and $X.Y \equiv \pm 1 \pmod 3$. This is consistent with the relations $S_i^2 = \mathbf{0}$ but $(S_i S_j)^2 \neq \mathbf{0}$.

The semigroup $\mathcal{S}$ exhibits a lot of symmetry. We find its automorphism group. Some automorphisms are inherited from the symmetry group Σ of the tetrahedron Δ, which has order 24 and is isomorphic to $\mathbb{S}_4$. Since Σ permutes the faces of Δ, it permutes the S_i by conjugation:

$$S_i \mapsto \sigma S_i \sigma^{-1}.$$

This action extends to $-S_i$ since conjugation commutes with the negative of the identity, and hence the action extends uniquely to any element of $\mathcal{S}$. Clearly this action defines automorphisms $\hat{\sigma}$ of $\mathcal{S}$, given by

$$\hat{\sigma}(S_i) = S_{\sigma(i)}\,, \quad \hat{\sigma}(-S_i) = -S_{\sigma(i)}\,, \quad \sigma \in \mathbb{S}_4\,, \qquad 0 \le i \le 3\,. \tag{4.1}$$

Define

$$\hat{\mathbb{S}}_4 = \left\{\hat{\sigma} \,:\, \sigma \in \mathbb{S}_4\right\}$$

which is isomorphic to $\mathbb{S}_4$. We now prove that the $\hat{\sigma}$ are the only automorphisms.

Proposition 4.4. *The automorphism group of $\mathcal{S}$ is the group $\hat{\mathbb{S}}_4$ with action* (4.1).

Proof. Suppose that α is an automorphism not in $\hat{\mathbb{S}}_4$. The elements in the subset $\mathcal{T} = \{\pm S_i : 1 \le i \le 4\}$ are the only nonzero elements with square $\mathbf{0}$, so any automorphism α permutes them. Moreover, within this set the annihilator of S_i is

$$\left\{U \in \mathcal{T} \,:\, US_i = \mathbf{0} = S_iU\right\} \;=\; \left\{S_i, -S_i \,:\, 1 \le i \le 4\right\}.$$

Therefore if α maps S_i to $\pm S_j$ then it must map $-S_i$ to $\mp S_j$. Composing with a suitable permutation in $\hat{\mathbb{S}}_4$ we can assume $\alpha(S_i) = \pm S_i$ for all i. Since $\alpha \notin \hat{\mathbb{S}}_4$, we have $\alpha(S_i) = -S_i$ for some i. For all $j \neq i$ the relation $S_iS_jS_i = S_i$ in (3.1) implies that $(-S_i)(\alpha(S_j))(-S_i) = -S_i$, so $\alpha(S_j) = -S_j$ for all j. But this contradicts the second relation $S_iS_jS_k = -S_iS_k$ in (3.1) which applies when all i, j, k are different. Thus every automorphism lies in $\hat{\mathbb{S}}_4$. ∎

Remark 4.5. Without using the above result, it is clear that the elements $\pm S_i$ satisfy $(\pm S_i)^2 = \mathbf{0}$. The products S_iS_j for $i \neq j$ are idempotent: $(S_iS_j)^2 = S_iS_j$. The elements $-S_iS_j$ for $i \neq j$ are not idempotent, but their squares are: $(-S_iS_j)^2 = S_iS_j$. Thus these three classes are distinguished by simple automorphism-invariant properties.

5. Density

We now use the classification of closed subgroups of $\mathbb{O}(3)$ to prove a density theorem for $\mathcal{R}$. This result is too weak to imply the result of [9] that almost closed non-embedded chains with arbitrarily small gaps exist, because it factors out the translations in the affine reflections, but it leads to the equidistribution theorem of Section 6 and is of independent interest. Here the notations $\mathbb{SO}(2), \mathbb{SO}(3), \mathbb{O}(3)$ refer to Lie groups defined over the real numbers. The full classification is not required explicitly; we just need a simple consequence:

LEMMA 5.1. *Let G be a closed subgroup of $\mathbb{O}(3)$. Then one of the following conditions is valid:*

(a) *G is finite.*

(b) *The subgroup of G consisting of elements with determinant $+1$ is conjugate to $\mathbb{SO}(2)$.*

(c) *$G = \mathbb{O}(3)$ or $G = \mathbb{SO}(3)$.*

Proof. This can be read off from the classification of closed subgroups of $\mathbb{O}(3)$; see for example [14, Theorem XIII.9.2]. ∎

THEOREM 5.2. *The group $\mathcal{R}$ is dense in $\mathbb{O}(3)$.*

Proof. Let $\overline{\mathcal{R}}$ be the closure of $\mathcal{R}$. This is a closed subgroup of $\mathbb{O}(3)$, so Lemma 5.1 applies. We consider the three cases in turn.

(a) This case does not apply: the group $\mathcal{R}$ is infinite since it is a free product.

(b) This case also does not apply. Since $\mathbb{SO}(2)$ is abelian, elements of $\overline{\mathcal{R}}$ with determinant 1 commute. In particular R_0R_1 and R_2R_3 commute. So

$$R_0R_1R_2R_3 = R_2R_3R_0R_1$$

contrary to Theorem 3.1.

(c) This is the only remaining case. The generators R_i do not belong to $\mathbb{SO}(3)$, so $\overline{\mathcal{R}} = \mathbb{O}(3)$. Therefore $\mathcal{R}$ is dense in $\mathbb{O}(3)$. ∎

COROLLARY 5.3. *The subgroup $\mathcal{R}^2 \subseteq \mathcal{R}$ generated by all products R_iR_j $(0 \leq i \neq j \leq 3)$ is dense in $\mathbb{SO}(3)$.*

Proof. Since $\det R_i = -1$, the subgroup $\mathcal{R}^2$ consists precisely of the elements of $\mathcal{R}$ that have determinant 1. Therefore $\mathcal{R}^2 = \mathcal{R} \cap \mathbb{SO}(3)$, which is dense in $\mathbb{SO}(3)$. ∎

6. Equidistribution

In this section we prove that the orbit of any point of the unit 2-sphere $\mathbb{S}^2$ for the action of the group $\mathcal{R}$ is equidistributed in $\mathbb{S}^2$, in a sense made precise in Definition 6.1 below. This is a natural analogue of the theorem of Arnold and Krylov [2], and we prove it using similar methods, including simplifications suggested by one reviewer that eliminate the use of spherical harmonics.

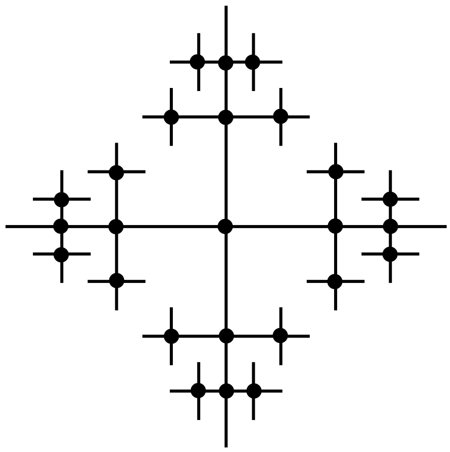

Figure 3: The Cayley graph of $\mathcal{R}$ (schematic). Dots (nodes) indicate group elements, with the identity being at the centre. Each edge represents left multiplication by a reflection R_i. Since $R_i^2 = I$ these edges are bidirectional. There are four types of edge, for $0 \leq i \leq 3$, and each node lies on one edge of each type. The tree structure continues recursively to infinity.

It is convenient to motivate the method in terms of the *Cayley graph* $\mathcal{C}(\mathcal{R})$ of $\mathcal{R}$, for the generating set $\{R_0, R_1, R_2, R_3\}$, see [5, 18]. The nodes of $\mathcal{C}(\mathcal{R})$ correspond to elements of $\mathcal{R}$. Edges (of type i) join node γ to $R_i\gamma$. The graph $\mathcal{C}(\mathcal{R})$ is an infinite tree, every node of which has valence 4, indicated schematically in Figure 3. Right multiplication by an element $\gamma \in \mathcal{R}$ induces an automorphism of $\mathcal{C}(\mathcal{R})$ that preserves edge types, because

$$R_i(\delta\gamma) = (R_i\delta)\gamma$$

for all $\delta \in \mathcal{R}$, so an edge of type i from δ to $R_i\delta$ maps to an edge of type i from $\delta\gamma$ to $R_i(\delta\gamma)$. In particular, $\mathcal{C}(\mathcal{R})$ is homogeneous in the sense that its automorphism group acts transitively. This is a hint that orbits might be equidistributed.

If $x \in \mathbb{S}^2$, the orbit $\mathcal{R}x$ wraps the nodes of $\mathcal{C}(\mathcal{R})$ around $\mathbb{S}^2$, sending $\gamma \in \mathcal{R}$ to γx. We can therefore use the structure of $\mathcal{C}(\mathcal{R})$ to represent the orbit. The length n of a product of reflections $R_{i_n} R_{i_{n-1}} \cdots R_{i_1} = \gamma \in \mathcal{R}$ corresponds to the length of a path in $\mathcal{C}(\mathcal{R})$ from the identity to that element. Such paths may intersect themselves, or repeat edges.

Consider a random walk on the Cayley graph of $\mathcal{R}$, where each edge of type i occurs with equal probability $\frac{1}{4}$ for $0 \leq i \leq 3$. After n steps the random walk reaches the group element

$$R_{i_n} R_{i_{n-1}} \cdots R_{i_1} \tag{6.1}$$

where successive R_{i_k} are chosen randomly from $\{0, 1, 2, 3\}$, each with probability $\frac{1}{4}$.

Let J_n be the set of all index sequences $\boldsymbol{j} = j_1, \ldots, j_n$, where $0 \leq j_k \leq 3$ for $k = 1, \ldots, n$. For $\boldsymbol{j} \in J_n$ define

$$R_{\boldsymbol{j}} = R_{j_n} R_{j_{n-1}} \cdots R_{j_1}$$

and let the length of the sequence be $\lambda(\boldsymbol{j}) = n$.

Assume that the random walk starts at the identity of $\mathcal{R}$. Let U be an open (or, more generally, measurable) subset of $\mathbb{S}^2$. For fixed but arbitrary $x \in \mathbb{S}^2$, let $P_n(U)$ be the probability that after n steps of the random walk the point

$$R_{i_n} R_{i_{n-1}} \cdots R_{i_1} x$$

belongs to U. Let μ be normalised surface Lebesgue measure on $\mathbb{S}^2$, that is, Lebesgue measure divided by 4π, so that $\mu(\mathbb{S}^2) = 1$. Heuristically, the orbit of x is equidistributed provided that

$$\lim_{n \to \infty} P_n(U) = \mu(U) \tag{6.2}$$

for all U. This motivates the following discussion, leading to the definition of equidistribution that we employ in this paper, Definition 6.1.

Let $\mathcal{C}(\mathbb{S}^2)$ be the space of all continuous maps $f : \mathbb{S}^2 \to \mathbb{R}$ with inner product

$$\langle f, g \rangle = \int_{\mathbb{S}^2} f g \, \mathrm{d}\mu \, .$$

The integral is finite because $\mathbb{S}^2$ is compact, and the inner product gives $\mathcal{C}(\mathbb{S}^2)$ a Hilbert space structure, inducing the norm

$$\|f\| = \sqrt{\langle f, f\rangle}\,. \tag{6.3}$$

The group $\mathbb{O}(3)$ of rotations in $\mathbb{R}^3$ has a natural action as isometries (norm-preserving maps) of $\mathcal{C}(\mathbb{S}^2)$, defined as follows. Suppose that $\gamma \in \mathbb{O}(3)$. The natural action of $\mathbb{O}(3)$ on $\mathbb{R}^3$ leaves $\mathbb{S}^2$ invariant, so for each $\gamma \in \mathbb{O}(3)$ there is an operator $\mathfrak{G}_\gamma$ on $\mathcal{C}(\mathbb{S}^2)$ defined by

$$(\mathfrak{G}_\gamma f)(x) = f\big(\gamma^{-1}x\big)\,. \tag{6.4}$$

The inverse ensures that this is a left action: $\mathfrak{G}_{\gamma\delta} f = \mathfrak{G}_\gamma \mathfrak{G}_\delta f$. Since μ is $\mathbb{O}(3)$-invariant,

$$\|\mathfrak{G}_\gamma f\| = \|f\| \tag{6.5}$$

for all $\gamma \in \mathbb{O}(3)$. Therefore every $\mathfrak{G}_\gamma$ is an isometry of $\mathcal{C}(\mathbb{S}^2)$.

In particular, there are isometries $\mathfrak{R}_i$ $(0 \le i \le 3)$ of $\mathcal{C}(\mathbb{S}^2)$ such that

$$(\mathfrak{R}_i f)(x) = f(R_i^{-1}x) = f(R_i x)$$

where the latter equality follows from $R_i^2 = I$. If we define

$$\overline{\mathfrak{R}} = \frac{\mathfrak{R}_0 + \mathfrak{R}_1 + \mathfrak{R}_2 + \mathfrak{R}_3}{4}$$

then the powers $\overline{\mathfrak{R}}^n$ correspond bijectively to paths of length n through the Cayley graph, weighted by the probability 4^{-n} of each such path. This motivates the following definition:

Definition 6.1. Let $x \in \mathbb{S}^2$, and let

$$f_n(x) = (\overline{\mathfrak{R}}^n f)(x) = \frac{1}{4^n}\sum_{\boldsymbol{j}\in J_n} f(R_{\boldsymbol{j}} x)$$

be the average value of f evaluated at images of x under elements of $\mathcal{R}$ having length n (corresponding to paths of length n in the Cayley graph). Then the $\mathcal{R}$-orbit of $x \in \mathbb{S}^2$ is *equidistributed* if and only if

$$\lim_{n\to\infty} f_n(x) = \int_{\mathbb{S}^2} f\,\mathrm{d}\mu \tag{6.6}$$

for any $f \in \mathcal{C}(\mathbb{S}^2)$.

Recall that the norm of a linear operator A on a Banach space $\mathcal{B}$ is defined by

$$\|A\| = \sup_{\|x\|=1} \|Ax\| \qquad (x \in \mathcal{B})\,.$$

Immediate consequences are:

$$\|Ax\| \leq \|A\|\|x\|\,, \qquad \|AB\| \leq \|A\|\|B\|\,. \tag{6.7}$$

As in Arnold and Krylov [2], we also need the following lemma:

LEMMA 6.2. *Let $v_1, \ldots, v_s \in \mathcal{C}(\mathbb{S}^2)$. Suppose that $\|v_i\| \neq 0$ for $1 \leq i \leq s$, and*

$$\|v_1 + \cdots + v_s\| = \|v_1\| + \cdots + \|v_s\|\,. \tag{6.8}$$

Then there exists $v \in \mathcal{C}(\mathbb{S}^2)$ and positive real numbers r_i $(1 \leq i \leq s)$ such that $v_i = r_i v$.

Proof. The triangle inequality implies that

$$\|v_1 + \cdots + v_s\| \;\leq\; \|v_1\| + \cdots + \|v_s\|$$

for any $v_i \in \mathcal{C}(\mathbb{S}^2)$. We claim that the conditions of the lemma imply that this is an equality with the stated properties of the v_i. To prove the claim, recall that a normed vector space is strictly convex if $x, y \neq 0$ and $\|x + y\| = \|x\| + \|y\|$ imply that $x = cy$ for some real constant $c > 0$. The space $\mathcal{C}(\mathbb{S}^2)$ is a Hilbert space, hence strictly convex [16, 6]. For such spaces, equality occurs in the triangle inequality if and only if all v_i are multiples of each other by nonnegative real numbers. A simple induction completes the proof. ∎

COROLLARY 6.3. *If $\|v_i\| = 1$ for $1 \leq i \leq s$, and* (6.8) *holds, then*

$$v_1 = \cdots = v_s\,. \tag{6.9}$$

We can now state and prove the main theorem of this section:

THEOREM 6.4. *If $x \in \mathbb{S}^2$ then the orbit $\mathcal{R}x$ is equidistributed in the sense of* (6.6).

Proof. Define a polynomial function in $\mathcal{C}(\mathbb{S}^2)$ to be the restriction to $\mathbb{S}^2$ of a polynomial function $\mathbb{R}^3 \to \mathbb{R}$ in Cartesian coordinates (x, y, z). By the Stone-Weierstrass Theorem [20, 22, 23, 29], polynomial functions are dense in $\mathcal{C}(\mathbb{S}^2)$ with the topology of uniform convergence. Therefore it is enough to prove (6.6) when f is polynomial.

This equality is obvious when f is constant, so it suffices to prove that $\lim(\overline{\mathfrak{R}}^n f)(x) = 0$ for any polynomial function whose integral over $\mathbb{S}^2$ is zero.

Let P_l be the vector space of polynomial functions $p : \mathbb{S}^2 \to \mathbb{R}$ of degree $\leq l$ such that $\int_{\mathbb{S}^2} p \, \mathrm{d}\mu = 0$. Since this space is finite-dimensional, any two Hausdorff vector topologies coincide, so the topology of pointwise convergence is the same as as that given by the norm (6.3). It is therefore sufficient to show that

$$\lim \|\overline{\mathfrak{R}}^n f\| = 0$$

for all $f \in P_l$.

Consider the linear operators $\overline{\mathfrak{R}}^n : P_l \to P_l$. We have $\|\mathfrak{R}_i f\| = \|f\|$, since reflections preserve μ. Hence $\|\overline{\mathfrak{R}}\| \leq 1$, so $\|\overline{\mathfrak{R}}^n\| \leq 1$ for any $n \in \mathbb{N}$.

Suppose that some $\|\overline{\mathfrak{R}}^m\| = k < 1$ where $m \geq 1$. Then (6.7) implies that

$$\|\overline{\mathfrak{R}}^n f\| \leq k^{\lfloor n/m \rfloor}.$$

Therefore $\lim_{n\to\infty} f_n(x) = 0$, proving (6.6).

Otherwise we must have $\|\overline{\mathfrak{R}}^n\| = 1$ for all $n \in \mathbb{N}$. We will show that this cannot occur. For a contradiction, suppose it does. The unit ball of P_l is compact, so there exists $f_n \in P_l$ with $\|\overline{\mathfrak{R}}^n f_n\| = \|f_n\| = 1$; hence also $\|\overline{\mathfrak{R}}^i f_n\| = 1$ for all $i \leq n$. A subsequence of the f_n converges to a polynomial $f \in P_l$ with $\|\overline{\mathfrak{R}}^n f\| = \|f\| = 1$ for all n.

Now Lemma 6.2, with the v_i being all $R_{\boldsymbol{j}} f$ for $\boldsymbol{j} \in J_n$, implies that

$$R_0 f = R_1 f = R_2 f = R_3 f\,, \quad R_0R_1 f = R_0R_2 f = \cdots = R_2R_3 f\,, \quad \ldots$$

and more generally,

$$R_{\boldsymbol{j}} f = R_{\boldsymbol{k}} f \qquad \text{for all } \boldsymbol{j}, \boldsymbol{k} \in J_n\,. \tag{6.10}$$

Let

$$K_m = \{R_{\boldsymbol{j}} \,:\, \lambda(\boldsymbol{j}) = m\}$$

be the set of all words in the R_i of length m, allowing consecutive repetitions. Since $R_i^2 = I$, we have

$$K_0 \subseteq K_2 \subseteq K_4 \subseteq \cdots \subseteq K_{2n} \subseteq \cdots\,.$$

Applying these group elements to $x \in \mathbb{S}^2$,

$$K_0 x \subseteq K_2 x \subseteq K_4 x \subseteq \cdots \subseteq K_{2n} x \subseteq \cdots\,.$$

Therefore the orbit of x under $\mathcal{R}^2 \subseteq \mathbb{SO}(3)$ is the union

$$\mathcal{R}^2 x = \bigcup_{n \in \mathbb{N}} K_{2n} x \,.$$

Now (6.10) implies that

$$f\big(R_{\boldsymbol{j}}^{-1} x\big) = f\big(R_{\boldsymbol{k}}^{-1} x\big)$$

for all $R_{\boldsymbol{j}}$, $R_{\boldsymbol{k}} \in K_{2n}$ and all $n \in \mathbb{N}$. Thus f is constant on the $\mathcal{R}^2$-orbit of x. By Corollary 5.3, $\mathcal{R}^2$ is dense in $\mathbb{SO}(3)$, so by continuity f is constant on the $\mathbb{SO}(3)$-orbit of x. By definition, elements of P_l have zero integral over $\mathbb{S}^2$, so this contradicts $\|f\| = 1$. ∎

Remark 6.5. (a) It is also plausible that $\mathcal{R}$ is equidistributed in $\mathbb{O}(3)$ with respect to Haar measure [15]. Equivalently, $\mathcal{R}^2$ is equidistributed in $\mathbb{SO}(3)$. However, we have not sought a proof.

(b) A similar argument shows that if $\mathcal{G}$ is a subgroup of $\mathbb{O}(n)$ generated by finitely many reflections, and the orbit $\mathcal{G}x$ of some point x in the unit sphere $\mathbb{S}^{n-1}$ is dense, then all orbits are equidistributed in $\mathbb{S}^{n-1}$ in the sense of the obvious generalisation of Definition 6.1.

Acknowledgements

I am grateful to Stan Wagon and Staszek Janeczko for discussions about the fascinating mathematics associated with chains of tetrahedra, and for helpful comments on a draft of this paper. I thank the two anonymous reviewers, who suggested numerous improvements, especially to the proof of Theorem 6.4. The presentation also benefited from a lecture given to the Warwick University Mathematics Society in November 2018, which motivated a closer examination of some subtle features of the proofs and removed a number of typographical errors.

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