Sylow 2-Subgroups of Solvable $\mathbb{Q}$-Groups

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Abstract: A finite group whose irreducible characters are rational valued is called a rational or a $\mathbb{Q}$-group. In this paper we obtain various results concerning the structure of a Sylow 2-subgroup of a solvable $\mathbb{Q}$-group.

Key words: $\mathbb{Q}$-groups, Sylow subgroups, extraspecial 2-groups.


1. INTRODUCTION AND PRELIMINARY RESULTS

Let $G$ be a finite group and $\chi$ be a complex character of $G$. Let $\mathbb{Q}(\chi)$ denote the subfield of the complex number $\mathbb{C}$ generated by all the values $\chi(x)$, $x \in G$. By definition $\chi$ is called rational if $\mathbb{Q}(\chi) = \mathbb{Q}$, where $\mathbb{Q}$ denotes the field of rational numbers. A finite group $G$ is called a rational group or a $\mathbb{Q}$-group, if every complex irreducible character of $G$ is rational. Equivalently $G$ is a $\mathbb{Q}$-group if and only if every $x$ in $G$ is conjugate to $x^m$ where $m \in \mathbb{N}$ is prime to the order of $x$. This will imply that for every $x \in G$ of order $n$ we have $N_G(\langle x \rangle) / C_G(\langle x \rangle) \cong \text{Aut}(\langle x \rangle)$ a group of order $\varphi(n)$ where $\varphi$ is the Euler $\varphi$-function. The symmetric group $S_n$ and the Weyl group of the complex Lie algebras are examples of $\mathbb{Q}$-groups. Elementary abelian 2-groups and extra-special 2-groups are also $\mathbb{Q}$-groups. Rational groups have been studied extensively, but their classification is far from being complete. It is proved in [5] that if $G$ is a solvable $\mathbb{Q}$-group, then $\pi(G) \subseteq \{2, 3, 5\}$. Also by [4] non-abelian composition factors of any finite $\mathbb{Q}$-group can only be either $SP_6(2)$ or $O^+_8(2)$. In [2] the structure of Frobenius $\mathbb{Q}$-groups has been found.

An important problem concerning $\mathbb{Q}$-groups is to classify them through the structure of a Sylow 2-subgroup. Any non-trivial $\mathbb{Q}$-group is of even order and there is a long standing conjecture that a Sylow 2-subgroup of a $\mathbb{Q}$-group is also a $\mathbb{Q}$-group [9, page 13]. The following results determine the structure of $\mathbb{Q}$-groups with a specified Sylow 2-subgroup.
Result 1. ([9, page 21] and [1, page 60]) Let $G$ be a $\mathbb{Q}$-group with an abelian Sylow 2-subgroup $P$. Then $P$ is an elementary abelian 2-group and $G$ is a supersolvable $\{2,3\}$-group. Moreover the commutator subgroup $G'$ of $G$ is a normal Sylow 3-subgroup of $G$ and $G$ splits over $G'$ with $P$ as a complement. In other words $G$ is a 2-nilpotent group.

Let $D_{2n}$ denote the dihedral group of order $2n$. It is proved in [9, page 25] that if $G$ is a $\mathbb{Q}$-group with a Sylow 2-subgroup $P$ isomorphic to $D_{2n}$, then $n = 1, 2$ or 4. If $n = 1$ or 2, then $P$ is abelian and the structure of $G$ follows from Result 1. But as far as the authors know the structure of $G$ in case $n = 4$ is not mentioned anywhere. However in [1, page 61] it is proved that if $G$ is a solvable group with a Sylow 2-subgroup isomorphic to $D_8$, then $\pi(G) \subseteq \{2,3\}$.

Result 2. ([9, page 35] and [1, page 62]) Let $G$ be a $\mathbb{Q}$-group with a Sylow 2-subgroup $P$ isomorphic to the quaternion group $Q_8$. Then $G$ contains a normal elementary abelian $p$-group $E_p$, where $p = 3$ or 5, and $G = E_p : P$, where " : " denotes semi-direct product. In other words $G$ is a 2-nilpotent group. Moreover if $G$ is non-nilpotent, then $G$ is isomorphic to a Frobenius group with complement isomorphic to $Q_8$, the quaternion group of order 8.

Motivated by the above results in this paper we obtain some properties of $\mathbb{Q}$-groups having certain Sylow 2-subgroups. We also determine when extensions of certain groups is a $\mathbb{Q}$-group. Finally we find conditions on solvable $\mathbb{Q}$-groups having an extra-special Sylow 2-subgroup.

2. SYLOW 2-SUBGROUPS OF $\mathbb{Q}$-GROUPS

In this section we study $\mathbb{Q}$-groups with certain conditions on their Sylow 2-subgroups.

Lemma 1. If a generalized quaternion group $P$ is the Sylow 2-subgroup of a $\mathbb{Q}$-group $G$, then $P$ is isomorphic to the quaternion group of order 8.

Proof. The generalized quaternion group of order $2^{n+1}$, $n \geq 2$, has the following presentation: $P = \langle x, y : x^{2^n} = 1, y^2 = x^{2^{n-1}}, y^{-1}xy = x^{-1} \rangle$. Suppose $G$ is a $\mathbb{Q}$-group and $P$ is a Sylow 2-subgroup of $G$. By definition we have $[N_G(\langle x \rangle) : C_G(\langle x \rangle)] = 2^n - 1$. Therefore $|N_G(\langle x \rangle)| = 2^{n-1} \times |C_G(\langle x \rangle)|$, hence the 2-part of $|N_G(\langle x \rangle)|$ is at least $2^{n-1} \times 2^n = 2^{2n-1}$. Since a Sylow 2-subgroup of $G$ has order $2^{n+1}$, we must have $2^{2n-1} \leq 2^{n+1}$, hence $n \leq 2$. Thus $|P| = 8$ and $P \cong Q_8$ is the quaternion group of order 8. \[\square\]
Proposition 1. Let $G$ be a solvable $\mathbb{Q}$-group of even order with exactly one conjugacy class of involutions. Then a Sylow 2-subgroup of $G$ is either elementary abelian or isomorphic to the quaternion group of order 8.

Proof. Let $S$ be a Sylow 2-subgroup of $G$. By [9] the center $Z(S)$ of $S$ is a non-trivial elementary abelian 2-group. If $x$ and $y$ are involutions in $Z(S)$, then by assumption $x$ and $y$ are conjugate in $G$. By a well-known result [10, page 137], $x$ and $y$ are conjugate in $N_G(S)$ the normalizer of $S$ in $G$. But by [9] we have $N_G(S) = S$. Therefore $x$ and $y$ are conjugate in $S$ implying $x = y$. Hence $|Z(S)| = 2$. Now assume $|S| > 2$. By a result of J. Thompson cited in [8, page 511], $S$ is isomorphic to a homocyclic or a Suzuki 2-group. If $S$ is homocyclic then $S$ is isomorphic to the direct product of cyclic groups of the same order, hence $Z(S) = S$ must be an elementary abelian 2-group. Otherwise if $S$ is a Suzuki 2-group, then by [8, page 311], $S' = \phi(S) = Z(S) = \{ x : x \in S, x^2 = 1 \}$, implying that $S$ has only one involution. Therefore $S$ must be isomorphic to a generalized quaternion group. Since $G$ is assumed to be a $\mathbb{Q}$-group, hence, by Lemma 1, $S$ is isomorphic to the quaternion group of order 8 and the proposition is proved.

Proposition 2. Let $G$ be a supersolvable $\mathbb{Q}$-group. Then Sylow 2-subgroups of $G$ are $\mathbb{Q}$-groups.

Proof. Let $G$ be a non-trivial supersolvable $\mathbb{Q}$-group. Then there is a cyclic normal subgroup $\langle x \rangle$ of prime order $p$ in $G$ where $p$ is the largest prime in $\pi(G)$. Now $N_G(x)/c_G(x) = \frac{G}{c_G(x)} \cong \mathbb{Z}_{p-1}$ is a $\mathbb{Q}$-group, hence $p - 1 \leq 2$. Therefore $\pi(G) \subseteq \{2, 3\}$. By [10, page 158], if $3 \mid |G|$ then a Sylow 3-subgroup $P$ of $G$ is normal in $G$. Hence $\frac{G}{P}$ is a Sylow 2-subgroup of $G$ which must be a $\mathbb{Q}$-group.

3. Extensions of abelian groups as $\mathbb{Q}$-groups

In this section we will consider split extensions of groups and determine when they are $\mathbb{Q}$-groups. Let a group $G$ act on a group $H$. The Cartesian product $H \times G$ endowed with the following law of composition: $(g, h)(g', h') = (gg', h^gh'^{-1})$, $g, g' \in G$, $h, h' \in H$, is a group called the semi-direct product of $H$ with $G$ and is denoted by $H \rtimes G$ or $H : G$. The group $L = H \rtimes G$ is also called a split extension of $H$ by $G$ and we may regard $H$ as a normal subgroup of $L$ such that $\frac{L}{H} \cong G$. 

**Lemma 2.** Split Extension of an elementary abelian 2-group by another elementary abelian 2-group is a \( \mathbb{Q} \)-group.

**Proof.** Let \( E_1 \) and \( E_2 \) be elementary abelian 2-groups and \( G = E_1 \rtimes E_2 \) be their semi-direct product. Operations of \( E_1 \) and \( E_2 \) will be written additively. Since \( \frac{G}{E_1} \cong E_2 \), every non-identity element of \( G \) is of order 2 or 4. To prove that \( G \) is a \( \mathbb{Q} \)-group it is enough to prove that every element of order 4 in \( G \) is conjugate to its inverse. Let \( x = (g, v) \in G \), where \( g \in E_2 \) and \( v \in E_1 \). If \( x \) is of order 4, then \( v + v^g \neq 0 \) and \((g, v)^{-1} = (g, v^g)\). Now \((1, v)^{-1}(g, v)(1, v) = (g, v^g)\), proving that \( x \) and \( x^{-1} \) are conjugate in \( G \) and the lemma is proved.

Let \( V \) be a vector space over a finite field on which the group \( G \) acts. Then we can form the usual semi-direct product \( V \rtimes G \) with the operation \((g, v)(h, u) = (gh, v^h + u)\), where \( g, h \in G \) and \( u, v \in V \). In the following we will assume \( G \) is a certain group and find necessary and sufficient conditions such that \( V \rtimes G \) is a \( \mathbb{Q} \)-group.

Let \( p \) be an odd prime and \( V \) be a 2-dimensional vector space over the Galois field \( GF(p) \). It is a well-known fact that there are \( a, b \in GF(p) \) such that \( a^2 + b^2 = -1 \). If we set

\[
i = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} -b & a \\ a & b \end{pmatrix},
\]

then it is easy to see that \( Q_8 = \langle i, j, k \rangle \) is isomorphic to the quaternion group of order 8. Therefore \( V \) is an irreducible module for \( Q_8 \) and we can form the semi-direct product \( V \rtimes Q_8 \). Our next result is the following.

**Proposition 3.** Let \( V \) be a 2-dimensional irreducible module over the field \( GF(p) \), \( p \) an odd prime, for the quaternion group \( Q_8 \). Then \( V \rtimes Q_8 \) is a \( \mathbb{Q} \)-group if and only if \( p = 3 \) or \( 5 \).

**Proof.** First we will prove that the order of elements of the group \( V \rtimes Q_8 \) is one of the numbers 1, 2, 4 or \( p \). Elements of \( V \rtimes Q_8 \) are of the forms \((g, v)\) where \( g \in Q_8 \) and \( v \in V \). It is obvious that for \( n \in N \) we have \((g, v)^n = (g^n, vg^{n-1} + v^2g^{n-2} + \cdots + v^n)\) and hence \( O(I, o) = 1, O(-I, o) = 2, O(x, v) = 4 \) for all \( x \in Q_8 - \{ \pm I \} \) and \( v \in V \), finally \( O(I, v) = p \) for all \( v \in V - \{0\} \). Now for elements \((g, v)\) and \((h, u)\) of \( V \rtimes Q_8 \) it can be verified that \((h, u)^{-1}(g, v)(h, u) = (h^{-1}gh, -uh^{-1}gh + vh + u)\).

Now if we consider \((x, v), v \in Q_8 - \{ \pm I \}, v \in V, \) then \( x \) and \( x^3 = -x \) are conjugate in \( Q_8 \) and hence there exists \( y \in Q_8 \) such that \( y^{-1}xy = -x \).
Therefore from \((y, u)^{-1}(x, v)(y, u) = (y^{-1}xy, -uy^{-1}xy + vy + u) = (-x, ux + vy + u) = (x, v)^3 = (x^3, vx^2 + vx + v) = (-x, vx)\) we will obtain \(ux + vy + u = vx\), thus \(u(x + I) = v(x - y)\) from which we will obtain \(u = \frac{1}{2}v(I + x + yx - y)\). Hence \((x, v)\) and \((x, v)^3\) for all \(x \in Q_8 - \{\pm I\}\) and \(v \in V\) are conjugate in \(V \rtimes Q_8\).

Now we will consider elements of order \(p\), say \((I, v)\), where \(v \in V - \{0\}\). Let \(m\) be an integer such that \(0 < m < p\) and \((x, u)^{-1}(I, v)(x, u) = (I, v)^m\), then \(vx = mv\). Hence \(m\) is an eigenvalue of \(x \in G\). But it is easy to see that eigenvalues of elements of \(Q_8\) are either \(\pm 1\) or roots of the equation \(t^2 + 1 = 0\) in \(GF(p)\). If the only eigenvalues occurring are \(\pm 1\), then \(p = 3\) and if roots of \(t^2 + 1 = 0\) occur we must have \(p = 5\). The converse is obviously true, i.e., if \(p = 3\) or \(5\), then \((I, v)\) is conjugate to \((I, v)^m\) for all \(0 < m < p\). The proposition is proved now.

Next we consider the symmetric group \(S_n\) of degree \(n\). In this case we assume \(V\) is a vector space of dimension \(n\) over the Galois field \(GF(q)\) where \(q\) is a power of the prime \(p\). We assume \(S_n\) as the symmetric group of the set \(\{1, 2, \ldots, n\}\) and \(V\) has basis \(\{e_1, \ldots, e_n\}\). Therefore the action of \(S_n\) on \(V\) is as follows: \(e_i\pi = e_{(i)\pi}\) for all \(1 \leq i \leq n\) and \(\pi \in S_n\). We consider the semi-direct product \(V \rtimes S_n\) called the hyperoctahedral group and prove the following result.

**Proposition 4.** \(V \rtimes S_n\) is a \(Q\)-group if and only if \(p = 2\).

**Proof.** With regard to the above explanation we consider the element \((1, e_i), 1 \leq i \leq n\), of order \(p\) in \(V \rtimes S_n\). This element must be conjugate to \((1, e_i)^m\), where \(0 < m < p\). Therefore there exists \((\pi, v) \in V \rtimes S_n\) such that \((\pi, v)^{-1}(1, e_i)(\pi, v) = (1, e_i)^m\) from which we obtain \(e_i\pi = me_i\) and therefore \(e_{(i)\pi} = me_i\) which implies \(m = 1\). Therefore \(p = 2\). By [9, Corollary 96A, page 96] the hyperoctahedral group \(B_n\) is a \(Q\)-group and this is the group \(V \rtimes S_n\) in the case \(p = 2\), the proof is complete now.

Now let \(V\) be a vector space of dimension \(n\) over the Galois field \(GF(q)\), \(q\) a power of the prime \(p\). Let \(G = GL_n(q)\) be the group of automorphisms of \(V\). Then \(G\) acts on \(V\) and we can form the semi-direct product \(V \rtimes G\). Our next result is concerned with the above consideration.

**Lemma 3.** Let \(q\) and \(n\) be positive integers. Then \(\varphi(q^n - 1) = n\) if and only if \((n, q) = (1, 2), (1, 3)\) or \((2, 2)\), where \(\varphi\) denotes the Euler \(\varphi\)-function.
Proof. If \( n = 1 \), then \( \varphi(q - 1) = 1 \) and obviously \( q - 1 = 1 \) or \( 2 \) implying \( q = 2 \) or \( 3 \). Therefore we will assume \( n \geq 2 \). It can be proved that for any positive integer \( m \) if \( q \geq 3 \), then \( q^m \geq m^2 \) and in the case of \( m \geq 4 \) we have \( 2^m \geq m^2 \). Now for any integer \( t \) it is easy to prove that \( \varphi(t) \geq \frac{1}{2} \sqrt{t} \). Hence if \( \varphi(q^n - 1) = n \), then \( n \geq \frac{1}{2} \sqrt{q^n - 1} \) which implies \( q = 2 \) or \( 3 \). Therefore we will assume \( n \geq 2 \). It can be proved that for any positive integer \( m \) if \( q \geq 3 \), then \( q^m \geq m^2 \) and in the case of \( m \geq 4 \) we have \( 2^m \geq m^2 \). Now for any integer \( t \) it is easy to prove that \( \varphi(t) \geq \frac{1}{2} \sqrt{t} \). Hence if \( \varphi(q^n - 1) = n \), then \( n \geq \frac{1}{2} \sqrt{q^n - 1} \) which implies \( q^2 < 2n + 1 \). First we assume \( n = 3 \) or \( 2 \). If \( n = 3 \), then \( q = 3 \) and if \( n = 2 \), then \( q = 3 \) or \( 4 \), and in both cases \( \varphi(q^n - 1) \neq n \). Now we will assume \( q = 2 \). If \( \frac{2}{2} - 1 \) implies \( n \leq 8 \). If \( n \geq 4 \), then from \( q^2 < 2n + 1 \) we obtain \( q = 2 \) which is not the case. Therefore \( n = 3 \) or \( 2 \). If \( n = 3 \), then \( q = 3 \) and if \( n = 2 \), then \( q = 3 \) or \( 4 \), and in both cases \( \varphi(q^n - 1) \neq n \). Now we will assume \( q = 2 \). If \( n \leq 8 \) we obtain \( \varphi(q^n - 1) = n \). Now by Lemma 3 we obtain \( (n, q) = (2, 2) \). The converse of the proposition is obvious and the Proposition is proved now.

Proposition 5. \( V \rtimes GL_n(q), n \geq 2, \) is a \( \mathbb{Q} \)-group if and only if \( (n, q) = (2, 2) \).

Proof. If \( H = V \rtimes GL_n(q) \) is a \( \mathbb{Q} \)-group, then by [9] the group \( \frac{H}{\mathbb{Z}^{(n)}} \cong GL_n(q) \) is also a \( \mathbb{Q} \)-group. Now for any \( \lambda \in GF(q)^* \) the matrices \( \lambda I \) and \( \lambda^{-1} I \) must be conjugate in \( GL_n(q) \) from which we will obtain \( \lambda^2 = 1 \) or \( \lambda = \pm 1 \). Therefore \( q = 2 \) or \( 3 \). Now by [7, page 187] the group \( GL_n(q) \) has an element \( h \) of order \( q^n - 1 \) such that \( \frac{N(h)}{GL_n(q)} \cong \mathbb{Z}_n \). Therefore \( \varphi(q^n - 1) = n \). Now by Lemma 3 we obtain \( (n, q) = (2, 2) \). The converse of the proposition is obvious and the Proposition is proved now.

4. Solvable \( \mathbb{Q} \)-groups with extraspecial Sylow 2-subgroup

As we mentioned in the introduction an extraspecial 2-group is a \( \mathbb{Q} \)-group and it may appear as a Sylow 2-subgroup. In of [1, problem 83, page 301] part 2 asks to classify rational \( \mathbb{Q} \)-groups with an extra-special Sylow 2-subgroup. Now we recall the definition of an extra-special \( p \)-group and its structure from [3].

Definition 1. A finite \( p \)-group \( P \) is called extra-special if \( P' = Z(P) \cong \mathbb{Z}_p \) and \( \frac{P}{P'} \) is an elementary abelian \( p \)-group.

Every extra-special \( p \)-group is the central product of non-abelian \( p \)-groups of order \( p^3 \). The dihedral group \( D_8 \) and the quaternion group \( Q_8 \) are extra-special 2-groups of order 8. If \( P \) is an extra-special 2-group, then there is an \( m \in \mathbb{N} \) such that \( |P| = 2^{2m+1} \). Moreover either \( P \cong D_8 \circ D_8 \circ \cdots \circ D_8 \) or
Let \( N \) be the pre-image of \( N \) in \( G \) and \( S \) be a Sylow 3-subgroup of \( G \). Then \( \bar{N} = N(x) \) and since \( x \) has order 2 we have \( x \notin S \). But \( x \in Z(G) \), hence \( x \in C_G(S) \) implying \( N = S \langle x \rangle \cong S \times \langle x \rangle \). Now \( S \) is a characteristic subgroup of \( N \) and hence \( S \trianglelefteq G \). Therefore \( S \leq O(G) = 1 \) which implies \( S = 1 \) and hence \( N = \langle x \rangle \). Consequently \( \bar{N} = 1 \) which gives the result \( G = P \) and the Lemma is proved.

**Proposition 6.** Let \( G \) be a \( \mathbb{Q} \)-group with an extra-special Sylow 2-subgroup \( P \). If \( Z(G) \neq 1 \), then \( G \) is a solvable group and there is a normal subgroup \( N \) of \( G \) with \( \pi(N) \subseteq \{3, 5\} \) such that \( G = NP \) and \( N \cap P = 1 \).

**Proof.** We use induction on \( O(G) \). If \( O(G) = 1 \), then by Lemma 4 we have \( G = P \) and \( N = 1 \) will work in the proposition. Therefore we may assume \( O(G) \neq 1 \). We know that \( \frac{G}{O(G)} \) is a \( \mathbb{Q} \)-group with a Sylow 2-subgroup isomorphic to \( P \). Since \( Z(G) \) is always an elementary abelian 2-group we obtain \( Z(G) \neq O(G) \) from which we deduce that \( Z \left( \frac{G}{O(G)} \right) \neq 1 \). Hence by induction we have \( \frac{G}{O(G)} \cong \bar{N}P \) where \( \bar{N} \subseteq \frac{G}{O(G)} \) and \( \bar{N} \cap P = 1 \). But \( \bar{N} = O \left( \frac{G}{O(G)} \right) = 1 \) and therefore \( G = O(G)P \). Now we set \( N = O(G) \), hence \( G = NP \). Since \( \frac{G}{N} \) is a solvable group and \( N \) has odd order we deduce that \( G \) is a solvable group. Now, by [5], \( G \) is a \( \{2, 3, 5\} \)-group and hence \( \pi(N) \subseteq \{3, 5\} \) and the proposition is proved.

Next we turn to solvable \( \mathbb{Q} \)-groups with an extra-special Sylow 2-subgroup. First of all let us determine the structure of the solvable \( \mathbb{Q} \)-groups with Sylow 2-subgroups isomorphic to the dihedral group \( D_8 \).

\( P \cong Q_8 \circ D_8 \circ \cdots \circ D_8 \), where \( \circ \) denotes the central product and in both cases \( m \) different groups are involved.

First we will prove the following two results about a general \( \mathbb{Q} \)-group. We recall that if \( G \) is a finite group, then the largest normal subgroup of odd order in \( G \) is denoted by \( O(G) \).

**Lemma 4.** Let \( G \) be a \( \mathbb{Q} \)-group with extra-special Sylow 2-subgroup \( P \). If \( G \) has a non-trivial center and \( O(G) = 1 \), then \( G = P \).

**Proof.** Since \( Z(G) \subseteq Z(P) = \langle x \rangle \) is a group of order 2 and \( Z(G) \) is assumed to be non-trivial, hence \( Z(G) = \langle x \rangle \). Now \( \frac{G}{(x)} \) is a \( \mathbb{Q} \)-group with \( \frac{P}{(x)} \) as a Sylow 2-subgroup. But \( \frac{P}{(x)} \) is an elementary abelian 2-group, hence, by Result 1, \( \frac{G}{(x)} \) is a supersolvable \( \{2, 3\} \)-group. Therefore there is a normal 3-subgroup \( \bar{N} \) of \( \frac{G}{(x)} \) such that \( \frac{G}{(x)} = \bar{N} \left( \frac{P}{(x)} \right) \). Let \( N \) be the pre-image of \( N \) in \( G \) and \( S \) be a Sylow 3-subgroup of \( G \). Then \( \bar{N} = \frac{N(x)}{(x)} \) and since \( x \) has order 2 we have \( x \notin S \). But \( x \in Z(G) \), hence \( x \in C_G(S) \) implying \( N = S \langle x \rangle \cong S \times \langle x \rangle \). Now \( S \) is a characteristic subgroup of \( N \) and hence \( S \trianglelefteq G \). Therefore \( S \leq O(G) = 1 \) which implies \( S = 1 \) and hence \( N = \langle x \rangle \). Consequently \( \bar{N} = 1 \) which gives the result \( G = P \) and the Lemma is proved.

\( P \cong Q_8 \circ D_8 \circ \cdots \circ D_8 \), where \( \circ \) denotes the central product and in both cases \( m \) different groups are involved.
Theorem 1. Let $G$ be a rational solvable group with a Sylow 2-subgroup isomorphic to $D_8$. Then $G$ contains a normal 3-subgroup $N$ such that $\frac{G}{N}$ is isomorphic to either $D_8$ or $S_4$.

Proof. By [1, page 61] we have $|G| = 8 \cdot 3^n$, where $n$ is a non-negative integer. The number of Sylow 3-subgroups $N_3$ of $G$ is either 1 or 4. If $N_3 = 1$, then a Sylow 3-subgroup $N$ of $G$ is normal in $G$ and $\frac{G}{N} \cong D_8$. Assume that $N_3 = 4$ and $\Omega = \{Q_1, Q_2, Q_3, Q_4\}$ is the set of distinct Sylow 3-subgroups of $G$. If $N$ denotes the kernel of the action of $G$ on $\Omega$ by conjugation, then $\frac{G}{N}$ is isomorphic to a subgroup of $S_4$. Since $G$ is assumed to be a $Q$-group, therefore $\frac{G}{N}$ is also a $Q$-group. Since $|N_G(Q_i)| = 2 \cdot 3^n$ and $N = \cap_{i=1}^4 N_G(Q_i)$, hence $4 \mid |\frac{G}{N}|$. Now it is easy to see that the rational subgroups of $S_4$ with order divisible by 4 are isomorphic to one of the groups $Z_2 \times Z_2$, $D_8$ or $S_4$.

If $\frac{G}{N} \cong Z_2 \times Z_2$, then $|N| = |N_G(Q_i)|$, for all $1 \leq i \leq 4$, which is a contradiction because the $Q_i$’s are distinct. If $\frac{G}{N} \cong D_8$ or $S_4$, then we are done and the Theorem is proved now.

Theorem 2. Let $G$ be a solvable $Q$-group with an extra-special Sylow 2-subgroup. Then one of the following possibilities holds:

(a) $G$ is a 2-nilpotent group.

(b) There is a proper normal subgroup $N$ of $G$ such that $\frac{G}{N} = P : E(2)$, where $P$ is a 3-group and $E(2)$ is an elementary abelian 2-group.

Proof. We use induction on $|G|$. Let $P$ be a Sylow 2-subgroup of $G$ which by assumption is extra-special. By [5] we have $\pi(G) \subseteq \{2, 3, 5\}$. Let $E$ be a minimal normal subgroup of $G$.

Case 1: $|E|$ is even. Therefore $E$ is a proper elementary abelian 2-subgroup of $G$ and we may assume $E \leq P$. Since $1 \neq E \leq P$, hence $E \cap Z(P) \neq 1$. But $Z(P) = P'$ is of order 2. Therefore $Z(P) = P' \subseteq E$. Thus $\frac{E}{P}$ is an abelian group and it is a Sylow 2-subgroup of $\frac{G}{E}$. Hence $\frac{G}{E}$ is a $Q$-group with an abelian Sylow 2-subgroup, hence by Result 1, $\frac{G}{E} = P : E(2)$ where $P$ is a 3-subgroup of $\frac{G}{E}$, and hence of $G$, and $E(2)$ is an elementary abelian 2-group. Therefore case (b) of the theorem holds.

Case 2: $|E|$ is odd. Hence $\frac{G}{E}$ is a $Q$-group with an extra-special Sylow 2-subgroup isomorphic to $P$.

If a minimal normal subgroup $\frac{A}{E}$ of $\frac{G}{E}$ has even order, then by Case 1, $(\frac{G}{E})/(\frac{A}{E}) \cong \frac{G}{A} = P : E(2)$ where $P$ is a 3-group and $E(2)$ is an elementary abelian 2-group as stated on part (b) of the theorem.
If a minimal normal subgroup $\frac{A}{E}$ of $G$ has an odd order, then $(\frac{G}{E})/(\frac{A}{E}) \cong \frac{G}{A}$, $|\frac{G}{E}| < |G|$ and $|A|$ is odd. Therefore by induction we reach a point such that there is a normal subgroup $N$ of $G$ with $\frac{G}{N}$ isomorphic to a Sylow 2-subgroup of $G$. This implies that $G$ is a 2-nilpotent group, and case (a) of the theorem holds.

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