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Essential g-Ascent and g-Descent of a Closed Linear Relation in Hilbert Spaces *

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Abstract: We define and discuss for a closed linear relation in a Hilbert space the notions of essential g-ascent (resp. g-descent) and g-ascent (resp. g-descent) spectrums. We improve in the Hilbert space case some results given by E. Chafai in a Banach space [Acta Mathematica Sinica, 34 B, 1212-1224, 2014] and several results related to the ascent (resp. essential ascent) spectrum for a bounded linear operator on a Banach space [Studia Math, 187, 59-73, 2008] are extended to closed linear relations on Hilbert spaces. We prove also a decomposition theorem for closed linear relations with finite essential g-ascent or g-descent.

Key words: Range subspace, Closed linear relation, Spectrum, Ascent, Essential ascent, Descent, Essential descent, Semi-Fredholm relation.

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1. Introduction and terminology

Let H be a complex Hilbert space. A multivalued linear operator $T : \mathsf{H} \longrightarrow \mathsf{H}$ or simply a linear relation is a mapping from a subspace $\mathcal{D}(T) \subseteq \mathsf{H}$, called the domain of T, into the collection of nonempty subsets of H such that $T(\lambda x + \mu y) = \lambda T(x) + \mu T(y)$ for all nonzero scalars λ, μ and x, $y \in \mathcal{D}(T)$. We denote by $\mathcal{L_R}(\mathsf{H})$ the class of linear relations on H. If T maps the points of its domain to singletons, then T is said to be a single valued linear operator or simply an operator. The graph $\mathsf{G}(T)$ of $T \in \mathcal{L_R}(\mathsf{H})$ is defined by :

$$\mathsf{G}(T) = \{(x, y) \in \mathsf{H} \times \mathsf{H} \,:\, x \in \mathcal{D}(T),\ y \in Tx\}.$$

We say that T is closed if its graph is a closed subspace of $\mathsf{H} \times \mathsf{H}$. The class of such linear relations will be denoted by $\mathcal{C_R}(\mathsf{H})$. A linear relation $T \in \mathcal{L_R}(\mathsf{H})$ is said to be continuous if for each open set $\Omega \subseteq \mathsf{Im}(T)$, $T^{-1}(\Omega)$ is an open

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set in $\mathcal{D}(T)$. Continuous everywhere defined linear relations are referred to as bounded relations. The kernel of a linear relation T is the subspace $\mathsf{ker}(T) := T^{-1}(0)$. The subspace $\mathsf{Im}(T) := T(\mathcal{D}(T))$ is called the range of T. The nullity and the defect of a linear relation $T \in \mathcal{L_R}(\mathsf{H})$ are defined by $\alpha(T) = \dim \mathsf{ker}(T)$ and $\beta(T) = \dim \mathsf{H}/\mathsf{Im}(T)$, respectively.

Recall that $T \in \mathcal{C_R}(\mathsf{H})$ is said to be *upper semi-Fredholm* if T has closed range and $\alpha(T) < +\infty$, and T is said to be *lower semi-Fredholm* if $\beta(T) < +\infty$. If T is upper or lower semi-Fredholm we say that T is *semi-Fredholm*, and we denote by $\Phi_\pm(\mathsf{H})$ the class of all semi-Fredholm relations. For $T \in \Phi_\pm(\mathsf{H})$ we define the *index* of T by

$$\mathrm{ind}(T) = \alpha(T) - \beta(T).$$

A linear relation is *Fredholm* if $\max\{\alpha(T), \beta(T)\} < +\infty$. We denote by $\Phi(\mathsf{H})$ (respectively, $\Phi_+(\mathsf{H})$, $\Phi_-(\mathsf{H})$) the class of all Fredholm (respectively, upper semi-Fredholm, lower semi-Fredholm) relations. The linear relation $T \in \mathcal{C_R}(\mathsf{H})$ is called regular if $\mathsf{Im}(T)$ is closed and $\mathsf{ker}(T) \subseteq \mathsf{Im}(T^n)$, for every $n \in \mathbb{N}$ (see [1]).

Recall that the resolvent set of $T \in \mathcal{L_R}(\mathsf{H})$ is defined (see, [4, Chapter VI]) by

$$\varrho(T) = \{\lambda \in \mathbb{C} \,:\, \lambda I - T \text{ is injective, open and has dense range}\}$$

and the spectrum of T is the set $\sigma(T) = \mathbb{C}\backslash\varrho(T)$. It is clear from the closed graph theorem for a linear relation that if T is a closed linear relation then $\varrho(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is bijective}\}$. We say that $T \in \mathcal{L_R}(\mathsf{H})$ has a trivial singular chain manifold if $\mathsf{R}_c(T) = \{0\}$ where

$$\mathsf{R}_c(T) = \Big[\bigcup_{i=1}^{\infty} \mathsf{ker}(T^i)\Big] \cap \Big[\bigcup_{i=1}^{\infty} T^i(0)\Big].$$

Let $\lambda \in \mathbb{C}$, by [10, Lemma 7.1], we know that $\mathsf{R}_c(T) = \{0\}$ if and only if $\mathsf{R}_c(\lambda I - T) = \{0\}$. It is easy to see that $\mathsf{R}_c(T) = \{0\}$ when $\varrho(T) \neq \emptyset$.

For two subspaces M and N of H, we recall that $\dim \mathsf{M}/\mathsf{M} \cap \mathsf{N} = \dim(\mathsf{M} + \mathsf{N})/\mathsf{N}$ and $(\mathsf{N} + \mathsf{W}) \cap \mathsf{M} = \mathsf{N} \cap \mathsf{M} + \mathsf{W}$ whenever W is a subspace of M.

Following [3], the *ascent* and the *descent* of $T \in \mathcal{L_R}(\mathsf{H})$ are respectively defined by

$$\boldsymbol{a}(T) = \inf\{k \in \mathbb{N} \,:\, \mathsf{ker}(T^{k+1}) = \mathsf{ker}(T^k)\},$$
$$\boldsymbol{d}(T) = \inf\{k \in \mathbb{N} \,:\, \mathsf{Im}(T^{k+1}) = \mathsf{Im}(T^k)\},$$

whenever these minima exist. If no such numbers exist the ascent and descent of T are defined to be $+\infty$.

For $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{H})$ and $n \in \mathbb{N}$, we define the following quantities :

$$\begin{array}{rcl} \alpha_n(T) & = & \dim \mathsf{ker}(T^{n+1})/\mathsf{ker}(T^n), \\ \beta_n(T) & = & \dim \mathsf{Im}(T^n)/\mathsf{Im}(T^{n+1}). \end{array}$$

Let us recall from [9, Lemma 3.2] and [10, Lemma 4.1], the following properties

$$\alpha_n(T) = \dim[\mathsf{Im}(T^n) \cap \mathsf{ker}(T)]/[T^n(0) \cap \mathsf{ker}(T)] \tag{1.1}$$

and

$$\begin{array}{rcl} \beta_n(T) & = & \dim \mathcal{D}(T^n)/[\mathsf{Im}(T) + \mathsf{ker}(T^n)] \cap \mathcal{D}(T^n) \\ & = & \dim[\mathsf{Im}(T) + \mathcal{D}(T^n)]/[\mathsf{Im}(T) + \mathsf{ker}(T^n)]. \end{array} \tag{1.2}$$

In [6], we show that $(\beta_n(T))_{n\geq 0}$ and $(\alpha_n(T))_{n\geq 0}$ are decreasing sequences. Recall that for $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{H})$, the *essential ascent*, $\boldsymbol{a_e}(T)$, and the *essential descent*, $\boldsymbol{d_e}(T)$, are defined by ([3])

$$\boldsymbol{a_e}(T) = \inf\{n \in \mathbb{N} : \alpha_n(T) < +\infty\},$$

$$\boldsymbol{d_e}(T) = \inf\{n \in \mathbb{N} :\ \beta_n(T) < +\infty\},$$

where the infimum over the empty set is taken to be infinite.

For $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{H})$ we consider the two decreasing sequences

$$\widetilde{\alpha}_n(T) = \dim \mathsf{Im}(T^n) \cap \mathsf{ker}(T), \quad \widetilde{\beta}_n(T) = \dim \mathsf{H}/[\mathsf{Im}(T) + \mathsf{ker}(T^n)], \ \ n \in \mathbb{N}.$$

Remark 1.1. From the equalities (1.1) and (1.2), we see that $\alpha_n(T) \leq \widetilde{\alpha}_n(T)$ and $\beta_n(T) \leq \widetilde{\beta}_n(T)$, for all $n \in \mathbb{N}$. Observe that $\mathsf{ker}(T) \cap T^j(0) \subseteq \mathsf{R}_c(T)$, for all $j \in \mathbb{N}$. Thus, by equality (1.1) (resp. (1.2)), it follows that if $\mathsf{R}_c(T) = \{0\}$ (resp. $\mathcal{D}(T^i) + \mathsf{Im}(T) = \mathsf{H}$, for all $i \in \mathbb{N}$), so $\alpha_n(T) = \widetilde{\alpha}_n(T)$ (resp. $\beta_n(T) = \widetilde{\beta}_n(T)$), for all $n \in \mathbb{N}$.

The above remark leads to the introduction of a new concept of g-ascent (resp. essential g-ascent, g-descent, essential g-descent) for a linear relation.

DEFINITION 1.2. Let $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{H})$.

(i) The *g-ascent*, $\widetilde{\boldsymbol{a}}(T)$, of T is defined by

$$\widetilde{\boldsymbol{a}}(T) = \inf\{n \in \mathbb{N} :\ \widetilde{\alpha}_n(T) = 0\}.$$

(ii) The *essential g-ascent*, $\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T)$, of T is defined as

$$\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T) = \inf\{n \in \mathbb{N} : \widetilde{\alpha}_n(T) < +\infty\}.$$

(iii) The *g-descent*, $\widetilde{\boldsymbol{d}}(T)$, of T is defined by

$$\widetilde{\boldsymbol{d}}(T) = \inf\{n \in \mathbb{N} \,:\, \widetilde{\beta}_n(T) = 0\}.$$

(iv) The *essential g-descent*, $\widetilde{\boldsymbol{d}}_{\boldsymbol{e}}(T)$, of T is defined by

$$\widetilde{\boldsymbol{d}}_{\boldsymbol{e}}(T) = \inf\{n \in \mathbb{N} \,:\, \widetilde{\beta}_n(T) < +\infty\},$$

where as usual the infimum over the empty set is taken to be $+\infty$.

It is clear that $\boldsymbol{a}(T) \leq \widetilde{\boldsymbol{a}}(T)$ and $\boldsymbol{a}_{\boldsymbol{e}}(T) \leq \widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T)$ (resp. $\boldsymbol{d}(T) \leq \widetilde{\boldsymbol{d}}(T)$ and $\boldsymbol{d}_{\boldsymbol{e}}(T) \leq \widetilde{\boldsymbol{d}}_{\boldsymbol{e}}(T)$), and equality holds when $\mathsf{R}_c(T) = \{0\}$ (resp. $\mathcal{D}(T^i) + \mathsf{Im}(T) = \mathsf{H}$, for all $i \in \mathbb{N}$).

The notion of ascent (resp. descent, essential ascent, essential descent) of a linear operator was studied in several papers (see for examples [2, 5]). In recent years some work has been devoted to extend these concepts to the case of linear relations, (see [3, 6, 10]). In [3], many basic results related to the ascent (resp. descent, essential ascent, essential descent) spectrum of linear operators have been extended to linear relations (usually with additional conditions). In this context, we prove that the results in [3] related to the spectral mapping theorem of ascent and essential ascent (resp. descent and essential descent) spectrums of a closed linear relation everywhere defined such that $\varrho(T) \neq \emptyset$ (resp. $\dim T(0) < +\infty$) remain valid when $T \in \Upsilon(\mathsf{H})$ (see page 142) (resp. $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{H})$) and without the assumption that $\varrho(T) \neq \emptyset$ (resp. $\dim T(0) < +\infty$) and $\mathcal{D}(T) = \mathsf{H}$. In [2], the ascent spectrum and the essential ascent spectrum of a bounded operator acting in Banach spaces are introduced and studied. In this paper, we extend in the Hilbert space case these notions to multivalued linear operators. However, the techniques used in this work are different from those used in [2, 3, 5]. Our approach here is based in the concept of range subspaces of Hilbert spaces (see, [7]).

The paper is organized as follows. In the next section, we first established some algebraic lemmas that will be used throughout this work. In Sections 3, 4 and 5, we are interested in the spectral theory of closed linear relations in Hilbert spaces having a finite essential g-ascent or finite essential g-descent. For example, in Theorem 3.4, we show that a closed linear relation with finite essential g-ascent is stable under perturbations of the form λI, where $\lambda \in \mathbb{C}$. In Theorem 3.8, we study the spectrum boundary points of a closed linear relation with finite essential g-ascent. In Theorem 4.12 and Theorem 5.4, we prove, under some conditions, that the essential g-ascent spectrum and essential g-descent spectrum satisfy the polynomial version of the spectral mapping theorem for a closed linear relation. Finally, in Section 6, we prove that if $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$, then there exists $n \in \mathbb{N}$ such that $\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T) \leq n$ and $\mathsf{Im}(T) + \mathsf{ker}(T^n)$ is closed (resp. $\widetilde{\boldsymbol{d}}_{\boldsymbol{e}}(T) \leq n$ and $\mathsf{Im}(T^n) \cap \mathsf{ker}(T)$ is closed) if and only if there exist $d \in \mathbb{N}$ and two closed subspaces M and N such that :

(i) $\mathsf{H} = \mathsf{M} \dot{+} \mathsf{N}$;

(ii) $\mathsf{Im}(T^d) \subseteq \mathsf{M}$, $T(\mathsf{M}) \subseteq \mathsf{M}$, $\mathsf{N} \subseteq \mathsf{ker}(T^d)$ and, if $d > 0$, $\mathsf{N} \not\subset \mathsf{ker}(T^{d-1})$;

(iii) $\mathsf{G}(T) = [\mathsf{G}(T) \cap (\mathsf{M} \times \mathsf{M})] \dot{+} [\mathsf{G}(T) \cap (\mathsf{N} \times \mathsf{N})]$;

(iv) the restriction of T to M is both upper semi-Fredholm (resp. lower semi-Fredholm) and regular relation;

(v) if $A \in \mathcal{L}_{\mathcal{R}}(\mathsf{N})$ such that its graph is the subspace $\mathsf{G}(T) \cap (\mathsf{N} \times \mathsf{N})$, then A is a bounded operator everywhere defined and $\mathsf{G}(A^d) = \mathsf{N} \times \{0\}$.

2. Algebraic preliminaries

Throughout this paper the symbol $\dot{+}$ denotes the topological direct sum of closed subspaces in H, i.e., $\mathsf{X}_0 = \mathsf{X}_1 \dot{+} \mathsf{X}_2$ if the linear space $\mathsf{X}_0 = \mathsf{X}_1 + \mathsf{X}_2$ is closed and $\mathsf{X}_1 \cap \mathsf{X}_2 = \{0\}$.

Next we give an example of quantities introduced below.

Example 2.1.

(i) Let M be a subspace of H and define the linear relation T in H by :

$$\mathcal{D}(T) = \mathsf{M} \text{ and } T(x) = \mathsf{M}, \ \forall\, x \in \mathsf{M}.$$

Clearly we have

$$\mathsf{ker}(T^n) = \mathsf{Im}(T^n) = \mathsf{M}, \ \forall\, n \geq 1. \tag{1}$$

• Case 1 : if $\dim \mathsf{M} = +\infty$, from (1), we have

$$\boldsymbol{a}(T) = \boldsymbol{a}_{\boldsymbol{e}}(T) = 1 < \widetilde{\boldsymbol{a}}(T) = \widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T) = +\infty.$$

- Case 2 : if $0 < \dim \mathsf{M} < +\infty$, by (1), we deduce that

$$\boldsymbol{a_e}(T) = \widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T) = 0 < \boldsymbol{a}(T) = 1 < \widetilde{\boldsymbol{a}}(T) = +\infty.$$

- Case 3 : if $0 < \dim \mathsf{H}/\mathsf{M} < +\infty$, from (1), we obtain

$$\widetilde{\boldsymbol{d}}_{\boldsymbol{e}}(T) = \boldsymbol{d_e}(T) = 0 < \boldsymbol{d}(T) = 1 < \widetilde{\boldsymbol{d}}(T) = +\infty.$$

- Case 4 : if $\dim \mathsf{H}/\mathsf{M} = +\infty$, it follows from (1) that

$$\boldsymbol{d}(T) = \boldsymbol{d_e}(T) = 1 < \widetilde{\boldsymbol{d}}_{\boldsymbol{e}}(T) = \widetilde{\boldsymbol{d}}(T) = +\infty.$$

(ii) Let M and N be a pair of closed subspaces of H such that $\mathsf{H} = \mathsf{M} \dotplus \mathsf{N}$ and $\{0\} \subsetneq \mathsf{N} \subsetneq \mathsf{H}$. Let P be the linear projection with domain H, range N and kernel M and let $L = P^{-1}$, then

$$\mathcal{D}(L) = \mathsf{N} \text{ and } L(x) = x + \mathsf{M}, \ \ \forall\, x \in \mathsf{N}.$$

Since $(I - L)x = \mathsf{M}$, with $x \in \mathsf{N}$, it follows that $\mathsf{ker}[(I - L)^n] = \mathsf{N}$ and $\mathsf{Im}[(I - L)^n] = \mathsf{M}$, for all $n \geq 1$. Thus

$$\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(I - L) = \boldsymbol{a_e}(I - L) \leq \boldsymbol{a}(I - L) = \widetilde{\boldsymbol{a}}(I - L) = 1,$$
$$\widetilde{\boldsymbol{d}}_{\boldsymbol{e}}(I - L) = \boldsymbol{d_e}(I - L) \leq \boldsymbol{d}(I - L) = \widetilde{\boldsymbol{d}}(I - L) = 1.$$

In particular, if $\dim \mathsf{N} < +\infty$, then $\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(I-L) = \boldsymbol{a_e}(I-L) = \widetilde{\boldsymbol{d}}_{\boldsymbol{e}}(I-L) = \boldsymbol{d_e}(I - L) = 0$.

In this section, we prove some algebraic results of the theory of linear relations which are used to prove the main results in this work.

For $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{H})$, we consider the sequence

$$S_n(T) = \dim[\mathsf{Im}(T^n) \cap \mathsf{ker}(T)]/[\mathsf{Im}(T^{n+1}) \cap \mathsf{ker}(T)],$$

$n \in \mathbb{N}$. From [10, Lemma 4.2], we get

$$S_n(T) = \dim[\mathsf{Im}(T) + \mathsf{ker}(T^{n+1})]/[\mathsf{Im}(T) + \mathsf{ker}(T^n)], \ \ \forall\, n \in \mathbb{N}.$$

The degree of stable iteration, $p(T)$, of T is defined by

$$p(T) = \inf\{n \in \mathbb{N} : S_m(T) = 0, \ \forall\, m \geq n\},$$

where the infimum over the empty set is taken to be infinite.

The following lemma helps to characterize the relationship between the degree of stable iteration and both the finite essential ascent and essential g-ascent of linear relations.

LEMMA 2.2. *Let* $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{H})$.

(i) *If* $\boldsymbol{a_e}(T) < +\infty$, *then*

$$p(T) \leq \inf\{n \in \mathbb{N} : \alpha_n(T) = \alpha_i(T), \ \forall\, i \geq n\} < +\infty.$$

(ii) *If* $\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T) < +\infty$ (*resp.* $\widetilde{\boldsymbol{a}}(T) < +\infty$), *then*

$$p(T) = \inf\{n \in \mathbb{N} : \widetilde{\alpha}_n(T) = \widetilde{\alpha}_m(T), \ \forall\, m \geq n\} < +\infty$$

and $\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T) \leq p(T)$ (*resp.* $\widetilde{\boldsymbol{a}}(T) = p(T)$).

Proof. (i) If $\boldsymbol{a_e}(T) < +\infty$, then $\alpha_n(T) < +\infty$, for every $n \geq \boldsymbol{a_e}(T)$. So, there exists $m \geq \boldsymbol{a_e}(T)$ such that $\alpha_n(T) = \alpha_m(T)$, for all $n \geq m$. Now by [6, equality (16)], we get $S_n(T) = 0$, for all $n \geq m$, and this proves that $p(T) \leq m$.

(ii) By [6, equality (18)], we can prove this assertion similarly as in (i), which completes the proof. ∎

Remark 2.3. Let $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{H})$ such that $\boldsymbol{a_e}(T) < +\infty$ (resp. $\boldsymbol{a}(T) < +\infty$). We note that $\boldsymbol{a_e}(T) \leq p(T)$ (resp. $\boldsymbol{a}(T) = p(T)$) in general is not true. Indeed, let T be defined as in Case 1 of Example 2.1, then $p(T) = 0 < \boldsymbol{a_e}(T) = \boldsymbol{a}(T) = 1$.

The next lemma exhibits some useful entirely algebraic properties of the degree of stable iteration of linear relations.

LEMMA 2.4. *Let* $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{H})$.

(i) *If* $\boldsymbol{d_e}(T) < +\infty$, *then*

$$p(T) \leq m = \inf\{n \in \mathbb{N} : \beta_n(T) = \beta_i(T), \ \forall\, i \geq n\} < +\infty.$$

(ii) *If* $\widetilde{\boldsymbol{d}}_{\boldsymbol{e}}(T) < +\infty$ (*resp.* $\widetilde{\boldsymbol{d}}(T) < +\infty$), *then*

$$p(T) = \inf\{n \in \mathbb{N} : \widetilde{\beta}_n(T) = \widetilde{\beta}_m(T), \ \forall\, m \geq n\} < +\infty$$

and $\widetilde{\boldsymbol{d}}_{\boldsymbol{e}}(T) \leq p(T)$ (*resp.* $\widetilde{\boldsymbol{d}}(T) = p(T)$).

Proof. Since

$$\mathsf{ker}(T^n)+\mathsf{Im}(T)\subseteq \mathsf{ker}(T^{n+1})+\mathsf{Im}(T)\subseteq \mathcal{D}(T^{n+1})+\mathsf{Im}(T)\subseteq \mathcal{D}(T^n)+\mathsf{Im}(T),$$

we deduce that

$$\beta_n(T)=\dim[\mathcal{D}(T^n)+\mathsf{Im}(T)]/[\mathcal{D}(T^{n+1})+\mathsf{Im}(T)]+\beta_{n+1}(T)+S_n(T). \quad (1)$$

But, since

$$\mathsf{Im}(T)+\mathsf{ker}(T^n)\subseteq \mathsf{Im}(T)+\mathsf{ker}(T^{n+1})\subseteq \mathsf{H},$$

it follows that

$$\widetilde{\beta}_n(T)=S_n(T)+\widetilde{\beta}_{n+1}(T). \quad (2)$$

Finally, the assertions (i) and (ii) follow from (1) and (2). The proof is therefore complete. ∎

Remark 2.5. Let $T\in\mathcal{L}_{\mathcal{R}}(\mathsf{H})$ such that $\boldsymbol{d_e}(T)<+\infty$ (resp. $\boldsymbol{d}(T)<+\infty$). We note that $\boldsymbol{d_e}(T)\le p(T)$ (resp. $\boldsymbol{d}(T)=p(T)$) in general is not true. Indeed, let T be defined as in Case 4 of Example 2.1, then $p(T)=0<\boldsymbol{d_e}(T)=\boldsymbol{d}(T)=1$.

3. Essential g-ascent spectrum and g-ascent spectrum of a closed relation

This section contains the main results of this work, in which we generalize some results of [2, Section 2] and our results of [5, Section 3] to the case of a closed linear relation in a Hilbert space H.

Throughout the remainder of the paper, for $T\in\mathcal{L}_{\mathcal{R}}(\mathsf{H})$ and $\lambda\in\mathbb{C}$, we denote by T_λ the relation $\lambda I-T$. The *ascent resolvent* set of $T\in\mathcal{C}_{\mathcal{R}}(\mathsf{H})$ is the set

$$\varrho_{asc}(T)=\{\lambda\in\mathbb{C}\,:\,\boldsymbol{a}(T_\lambda)<+\infty \text{ and } \mathsf{Im}(T_\lambda)+\mathsf{ker}[(T_\lambda)^{\boldsymbol{a}(T_\lambda)}] \text{ is closed}\}$$

and its *ascent spectrum*

$$\sigma_{asc}(T)=\mathbb{C}\backslash\varrho_{asc}(T).$$

The *essential ascent resolvent* and the *essential ascent spectrum* of $T\in\mathcal{C}_{\mathcal{R}}(\mathsf{H})$ are defined respectively by

$$\varrho^e_{asc}(T)=\{\lambda\in\mathbb{C}\,:\,\boldsymbol{a_e}(T_\lambda)<+\infty \text{ and } \mathsf{Im}(T_\lambda)+\mathsf{ker}[(T_\lambda)^{\boldsymbol{a_e}(T_\lambda)}] \text{ is closed}\}$$

and

$$\sigma^{e}_{asc}(T) = \mathbb{C}\backslash\varrho^{e}_{asc}(T).$$

The *g-ascent resolvent* set of $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$ is the set

$$\varrho^{g}_{asc}(T) = \{\lambda \in \mathbb{C} \,:\, \widetilde{\boldsymbol{a}}(T_\lambda) < +\infty \text{ and } \mathsf{Im}(T_\lambda) + \mathsf{ker}[(T_\lambda)^{\widetilde{\boldsymbol{a}}(T_\lambda)}] \text{ is closed}\}$$

and its *g-ascent spectrum*

$$\sigma^{g}_{asc}(T) = \mathbb{C}\backslash\varrho^{g}_{asc}(T).$$

The *essential g-ascent resolvent* and the *essential g-ascent spectrum* of $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$ are defined respectively by

$$\varrho^{e,g}_{asc}(T) = \{\lambda \in \mathbb{C} \,:\, \widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T_\lambda) < +\infty \text{ and } \mathsf{Im}(T_\lambda) + \mathsf{ker}[(T_\lambda)^{\widetilde{\boldsymbol{a}}_e(T_\lambda)}] \text{ is closed}\}$$

and

$$\sigma^{e,g}_{asc}(T) = \mathbb{C}\backslash\varrho^{e,g}_{asc}(T).$$

From [6, Lemma 2.9], we deduce easily the following

$$\varrho(T) \subseteq \varrho^{g}_{asc}(T) \subseteq \varrho_{asc}(T) \subseteq \varrho^{e}_{asc}(T),$$

$$\varrho(T) \subseteq \varrho^{g}_{asc}(T) \subseteq \varrho^{e,g}_{asc}(T) \subseteq \varrho^{e}_{asc}(T).$$

Let us recall the following definition.

DEFINITION 3.1. ([7], DEFINITION 3.3, DEFINITION 4.1)

(i) A subspace M of a Hilbert space H is said to be a range subspace of H if there exist a Hilbert space N and a bounded operator T from N to H such that $\mathsf{M} = \mathsf{Im}(T)$. In particular, a closed subspace of a Hilbert space H is a range subspace of H.

(ii) An operator or a relation $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{H})$ is said to be a range space operator or range space relation if its graph $\mathsf{G}(T)$ is a range subspace of $\mathsf{H} \times \mathsf{H}$.

It is clear that a closed relation in a Hilbert space H is a range space relation in H. Our approach here is based in the concept of range subspaces of Hilbert spaces (see, [7]).

We have the following lemma, which will be needed in the sequel.

Lemma 3.2. *Let $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$.*

(*i*) *If $\boldsymbol{a_e}(T) < +\infty$, $T^k(0)$ and $\mathsf{Im}(T) + \mathsf{ker}(T^{\boldsymbol{a_e}(T)})$ are closed for all $k \in \mathbb{N}$, then*

a) *$\mathsf{Im}(T^n)+\mathsf{ker}(T^m)$ is closed, for all $m+n \geq p(T)$. In particular $\mathsf{Im}(T^n)$ is closed for all $n \geq p(T)$.*

b) *$\mathsf{Im}(T^n) + \mathsf{ker}(T^m)$ is closed, for all $n \in \mathbb{N}$ and $m \geq \boldsymbol{a_e}(T)$.*

(*ii*) *If $\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T) < +\infty$ and $\mathsf{Im}(T) + \mathsf{ker}(T^{\widetilde{\boldsymbol{a}}_e(T)})$ is closed, then*

a) *$T^n(0)$ is closed, for all $n \in \mathbb{N}$, and so from [6, Lemma 2.9] and (i), $\mathsf{Im}(T^n) + \mathsf{ker}(T^m)$ is closed, for all $m + n \geq p(T)$ (resp. $n \in \mathbb{N}$ and $m \geq \boldsymbol{a_e}(T)$),*

b) *$\mathsf{ker}(T^n)$ and $\mathsf{Im}(T^{\boldsymbol{a}_e(T)+n})$ are closed, for all $n \in \mathbb{N}$.*

Proof. (*i*) From [6, equality (17)], we have $\mathsf{Im}(T^n) + \mathsf{ker}(T^m)$ is closed for all n, $m \in \mathbb{N}$ such that $m+n \geq p(T)$. In particular $\mathsf{Im}(T^n) + \mathsf{ker}(T^{p(T)+\boldsymbol{a}_e(T)})$ is closed for every $n \in \mathbb{N}$, so by [6, Lemma 2.9], $\mathsf{Im}(T^n) + \mathsf{ker}(T^m)$ is closed, for all $n \in \mathbb{N}$ and $m \geq \boldsymbol{a_e}(T)$.

(*ii*) From [6, Lemma 2.10], it follows that $T^n(0)$ and $\mathsf{Im}(T^{\boldsymbol{a}_e(T)+n})$ are closed, for all $n \in \mathbb{N}$.
Let $S = T_{|\mathsf{Im}(T^{p(T)})}$ be the restriction of T to $\mathsf{Im}(T^{p(T)})$. Now, let $n \in \mathbb{N}$, by Lemma 2.2, we have $\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T) \leq p(T)$, this implies that

$$\dim \mathsf{ker}(T^n) \cap \mathsf{Im}(T^{p(T)}) = \alpha(S^n) \leq n\,\alpha(S) = n\,\widetilde{\alpha}_{p(T)}(T) < +\infty$$

(see [9, Lemma 5.1]) and as $\mathsf{Im}(T^{p(T)})+\mathsf{ker}(T^n)$ is closed, so by [7, Propositions 3.9, 3.10, 4.8, Lemma 4.2], $\mathsf{ker}(T^n)$ is closed, for all $n \in \mathbb{N}$. This completes the proof of the lemma. ∎

For $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{H})$ and $k \in \mathbb{N}$, $\widetilde{T_k}$ denotes the following linear relation :

$$\begin{array}{cccl} \widetilde{T_k} : & \mathcal{D}(T)/\mathsf{ker}(T^k) \subseteq \mathsf{H}/\mathsf{ker}(T^k) & \longrightarrow & \mathsf{H}/\mathsf{ker}(T^k) \\ & \overline{x} & \longmapsto & \overline{Tx} := \{\overline{z} \, : \, z \in Tx\}. \end{array}$$

We will prove first that the linear relation $\widetilde{T_k}$ is well-defined. To do this, let us choose x_1, $x_2 \in \mathcal{D}(T)$ such that $x_1 - x_2 \in \mathsf{ker}(T^k)$. So, $0 \in T^k(x_1 - x_2)$, and therefore there exists $x \in T(x_1 - x_2) = Tx_1 - Tx_2$, such that $0 \in T^{k-1}(x)$.

From this, we get $x \in \mathsf{ker}(T^{k-1})$ and $\overline{0} \in \overline{Tx_1} - \overline{Tx_2}$. Let $y \in T(x_1)$, since $T(x_1) = y + T(0)$ then

$$\overline{0} \in \overline{y} + \overline{T(0)} - \overline{Tx_2} = \overline{y} - \overline{Tx_2}.$$

Hence, $\overline{T(x_1)} \subseteq \overline{T(x_2)}$ and by interchanging x_1 and x_2, we deduce that $\overline{T(x_1)} = \overline{T(x_2)}$.

Let M be a closed subspace of H, then H/M is a Hilbert space with the inner product

$$\begin{array}{rccl} <\cdot\,,\,\cdot>_{\mathsf{M}}: & \mathsf{H}/\mathsf{M} \times \mathsf{H}/\mathsf{M} & \longrightarrow & \mathbb{R} \\ & (\overline{x}\,,\,\overline{y}) & \longmapsto & <P(x)\,,\,P(y)>, \end{array}$$

where P is the orthogonal projection onto $\mathsf{M}^{\perp}$ and $<\cdot\,,\,\cdot>$ is the standard inner product on H. Note, the Hilbert space topology of $(\mathsf{H}/\mathsf{M}, <\cdot\,,\,\cdot>_{\mathsf{M}})$ coincides with the quotient topology of H/M :

$$\|\overline{x}\| = \sqrt{<\overline{x}, \overline{x}>_{\mathsf{M}}} = \sqrt{<P(x), P(x)>} = \mathrm{dist}(x, \mathsf{M}),$$

where $\mathrm{dist}(x, \mathsf{M})$ denotes, as usual, the distance of x to M. Now, let $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$ such that $\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T) < +\infty$ and $\mathsf{Im}(T) + \mathsf{ker}(T^{\widetilde{a}_e(T)})$ is closed. From Lemma 3.2, it follows easily that $\mathsf{H}/\mathsf{ker}(T^n)$ is a Hilbert space, for all $n \in \mathbb{N}$.

In [2, Lemma 2.1], it was established that if T is a bounded operator and T admits a finite essential g-ascent such that $\mathsf{Im}(T) + \mathsf{ker}(T^{\widetilde{a}_e(T)})$ is closed, then $\widetilde{T}_j$ is both regular and upper semi-Fredholm operator, for all $j \geq p(T)$, where $\widetilde{T}_j$ is the operator induced by T on $\mathsf{H}/\mathsf{ker}(T^j)$. In [5, Lemma 3.4], this result was extended to the case of unbounded closed operators, and in the following lemma, we prove that this result remains valid even in the context of closed linear relations.

Lemma 3.3. *Let $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$ such that $\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T) < +\infty$ and $j \geq p(T)$. If $\mathsf{Im}(T) + \mathsf{ker}(T^{\widetilde{a}_e(T)})$ is closed, then*

$$\begin{array}{rccl} \widetilde{T}\;: & \mathcal{D}(T)/\mathsf{ker}(T^j) \subseteq \mathsf{H}/\mathsf{ker}(T^j) & \longrightarrow & \mathsf{H}/\mathsf{ker}(T^j) \\ & \overline{x} & \longmapsto & \overline{Tx} := \{\overline{y}\;:\; y \in Tx\} \end{array}$$

is both regular and upper semi-Fredholm relation.

Proof. First, recall that from Lemma 3.2, we have $\mathsf{ker}(T^n)$ is closed for all $n \in \mathbb{N}$, and in particular, $\mathsf{H}/\mathsf{ker}(T^j)$ is a Hilbert space. Now, we will show that $\mathsf{ker}(\widetilde{T}) = \mathsf{ker}(T^{j+1})/\mathsf{ker}(T^j)$. To do this let $x \in \mathsf{ker}(T^{j+1})$, so $y \in Tx$ and $0 \in T^j y$ for some $y \in \mathsf{H}$. Since $0_{\mathsf{H}/\mathsf{ker}(T^j)} = \overline{y} \in \overline{Tx}$, it follows that $\mathsf{ker}(T^{j+1})/\mathsf{ker}(T^j) \subseteq \mathsf{ker}(\widetilde{T})$. In order to prove the converse inclusion, assume that $\overline{x} \in \mathsf{ker}(\widetilde{T})$. From $\overline{0} \in \overline{Tx}$, we deduce that $T(x) \cap \mathsf{ker}(T^j) \neq \emptyset$, which shows that $x \in T^{-1}(\mathsf{ker}(T^j)) = \mathsf{ker}(T^{j+1})$. Consequently $\mathsf{ker}(\widetilde{T}) \subseteq \mathsf{ker}(T^{j+1})/\mathsf{ker}(T^j)$. Moreover, it is clear that $\mathsf{Im}(\widetilde{T}) = [\mathsf{Im}(T) + \mathsf{ker}(T^j)]/\mathsf{ker}(T^j)$, so from Lemma 3.2, we get $\mathsf{Im}(\widetilde{T})$ is closed. Let π be the natural quotient map with domain $\mathsf{H}\times\mathsf{H}$ and null space $\mathsf{ker}(T^j)\times\mathsf{ker}(T^j)$, then $\mathsf{G}(\widetilde{T}) = \pi(\mathsf{G}(T))$. From [7, Corollary 4.9], we know that there exist a Hilbert space Z and a bounded operator θ from Z to $\mathsf{H} \times \mathsf{H}$ such that $\mathsf{Im}(\theta) = \mathsf{G}(T)$. This implies that $\pi\theta(\mathsf{Z}) = \mathsf{G}(\widetilde{T})$. Therefore $\widetilde{T}$ is a range space relation.

On the other hand, we have $\alpha(\widetilde{T}) = \alpha_j(T) < +\infty$ and $\mathsf{Im}(\widetilde{T})$ is closed. So from [7, Lemma 4.6], $\widetilde{T}$ is closed, and hence $\widetilde{T}$ is upper semi-Fredholm. Furthermore, by [7, Lemma 2.5], it follows that $\mathsf{ker}(T^{j+1}) \subseteq \mathsf{Im}(T^n)+\mathsf{ker}(T^j)$, for all $n \in \mathbb{N}$. Consequently, $\mathsf{ker}(\widetilde{T}) \subseteq \mathsf{Im}(\widetilde{T}^n)$, for all $n \in \mathbb{N}$, and the proof is therefore complete. ∎

Now, we are ready to state our main result of this section, which is an extension of [2, Theorem 2.3] and [5, Theorem 3.8] to closed multivalued linear operators.

Theorem 3.4. *Let $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$ such that $\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T) < +\infty$ and $\mathsf{Im}(T) + \mathsf{ker}(T^{\widetilde{\boldsymbol{a}}_e(T)})$ is closed. Then there is $\varepsilon > 0$ such that for every $0 < |\lambda| < \varepsilon$, we have :*

(*i*) *T_λ is both regular and upper semi-Fredholm,*

(*ii*) $\alpha_{p(T)}(T) \leq \alpha(T_\lambda) \leq (p(T)+1)\,\widetilde{\alpha}_{p(T)}(T)$,

(*iii*) $\beta(T_\lambda) = \dim \mathsf{H}/[\mathsf{Im}(T) + \mathsf{ker}(T^{p(T)})]$.

Proof. Let $k = p(T)$ and let $\widetilde{T_k}$ be the relation induced by T on $\mathsf{H}/\mathsf{ker}(T^k)$. First, from Lemma 3.3, $\widetilde{T_k}$ is both regular and upper semi-Fredholm relation. Using [1, Theorems 23, 25], we deduce that there exists $\varepsilon > 0$, such that $\lambda I - \widetilde{T_k}$ is both regular and upper semi-Fredholm relation for every $0 < |\lambda| < \varepsilon$. Furthermore, by [4, Corollary V.15.7] and [1, Theorem 27], we have

$$\alpha(\widetilde{T_k}) = \alpha(\lambda I - \widetilde{T_k}), \quad \beta(\widetilde{T_k}) = \beta(\lambda I - \widetilde{T_k}), \quad \forall\, \lambda \in \mathbb{C},\ 0 < |\lambda| < \varepsilon. \tag{1}$$

We prove first that

$$\mathsf{ker}(\lambda I - \widetilde{T_k}) = [\mathsf{ker}(\lambda I - T) + \mathsf{ker}(T^k)]/\mathsf{ker}(T^k), \tag{2}$$

$$\mathsf{Im}[(\lambda I-\widetilde{T_k})^n]=\mathsf{Im}[(\lambda I-T)^n]/\mathsf{ker}(T^k),\ \ \forall\, n\in\mathbb{N}. \tag{3}$$

If $\overline{x}\in \mathsf{ker}(\lambda I-\widetilde{T_k})$, then $\overline{(\lambda I-T)(0)}=\overline{(\lambda I-T)(x)}$, and so $T(0)+\mathsf{ker}(T^k)=(\lambda I-T)x+\mathsf{ker}(T^k)$. It is clear that $x\in\mathcal{D}(T^i)$, for all $i\in\mathbb{N}$ and $T(0)+T^{-k}(0)=(\lambda I-T)x+T^{-k}(0)$. From [8, Corollary 2.1] and [4, Corollary I.2.10], we obtain $T^{k+1}(0)+T^k(0)=(\lambda I-T)T^kx+T^k(0)$. Since $T^k(0)\subseteq T^{k+1}(0)=(\lambda I-T)T^k(0)$ (see [8, Theorem 3.6]), it follows that $(\lambda I-T)T^kx=(\lambda I-T)T^k(0)$, which implies that $x\in \mathsf{ker}[(\lambda I-T)T^k]$ and $\overline{x}\in\mathsf{ker}[(\lambda I-T)T^k]/\mathsf{ker}(T^k)$. Consequently $\mathsf{ker}(\lambda I-\widetilde{T_k})\subseteq \mathsf{ker}[(\lambda I-T)T^k]/\mathsf{ker}(T^k)$. To prove the converse inclusion, let $\overline{x}\in\mathsf{ker}[(\lambda I-T)T^k]/\mathsf{ker}(T^k)$, so $0\in T^k(\lambda I-T)x$. This implies that there exists $z\in\mathsf{H}$ such that $z\in(\lambda I-T)x$ and $0\in T^kz$. Hence $0=\overline{z}\in\overline{(\lambda I-T)x}$, and $\overline{x}\in\mathsf{ker}(\lambda I-\widetilde{T_k})$. Now from [8, Theorem 3.4], we obtain

$$\mathsf{ker}(\lambda I-\widetilde{T_k})=[\mathsf{ker}(\lambda I-T)+\mathsf{ker}(T^k)]/\mathsf{ker}(T^k).$$

Since $\mathsf{ker}(T^k)\subseteq\mathsf{Im}[(\lambda I-T)^n]$, it follows that

$$\mathsf{Im}[(\lambda I-\widetilde{T_k})^n]=[\mathsf{Im}[(\lambda I-T)^n]+\mathsf{ker}(T^k)]/\mathsf{ker}(T^k)=\mathsf{Im}[(\lambda I-T)^n]/\mathsf{ker}(T^k).$$

(i) Let $S=T_{|\mathsf{Im}(T^k)}$ be the restriction of T to $\mathsf{Im}(T^k)$. By (1) and (2), we obtain

$$\begin{aligned}\dim\mathsf{ker}(\lambda I-T)/\mathsf{ker}(\lambda I-T)\cap\mathsf{ker}(T^k)=&\dim[\mathsf{ker}(\lambda I-T)\\ &+\mathsf{ker}(T^k)]/\mathsf{ker}(T^k)\\ =&\alpha(\widetilde{T_k})<+\infty\end{aligned} \tag{4}$$

and

$$\begin{aligned}\dim\mathsf{ker}(\lambda I-T)\cap\mathsf{ker}(T^k)&\le\dim\mathsf{Im}(T^k)\cap\mathsf{ker}(T^k)\\ &=\alpha(S^k)\\ &\le k\alpha(S)=k\,\widetilde{\alpha}_k(T)<+\infty\end{aligned} \tag{5}$$

(see [9, Lemma 5.1]). In particular, this proves that $\alpha(\lambda I-T)<+\infty$. From (3), we infer that $\mathsf{Im}(\lambda I-T)$ is closed. Finally, since $\lambda I-\widetilde{T_k}$ is regular, by (2) and (3), we deduce that $\lambda I-T$ is regular.

(ii) From (4) and (5), we get

$$\begin{aligned}\alpha_k(T)=\alpha(\widetilde{T_k})&\le\alpha(\lambda I-T)=\alpha_k(T)+\dim\mathsf{ker}(\lambda I-T)\cap\mathsf{ker}(T^k)\\ &\le\widetilde{\alpha}_k(T)+k\,\widetilde{\alpha}_k(T)\le(k+1)\,\widetilde{\alpha}_k(T).\end{aligned}$$

Assertion (iii) follows from (1) and (3), which completes the proof. ∎

The following theorem is a simple consequence of Theorem 3.4.

THEOREM 3.5. *Let* $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$ *such that* $\widetilde{\boldsymbol{a}}(T) < +\infty$ *and* $\mathsf{Im}(T) + \mathsf{ker}(T^{\widetilde{\boldsymbol{a}}(T)})$ *is closed. Then there is* $\varepsilon > 0$ *such that for every* $0 < |\lambda| < \varepsilon$, *we have*

(*i*) T_λ *is injective with closed range,*

(*ii*) $\beta(T_\lambda) = \dim \mathsf{H}/[\mathsf{Im}(T) + \mathsf{ker}(T^{p(T)})]$.

COROLLARY 3.6. *Let* $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$, *then* $\sigma^{g}_{asc}(T)$ *and* $\sigma^{e,g}_{asc}(T)$ *are closed.*

For $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$, we consider the set :

$$\mathsf{E}(T) = \{\lambda \in \sigma(T) \,:\, \lambda \text{ an isolated point}, \widetilde{\boldsymbol{a}}(T_\lambda) < +\infty, \\ \widetilde{\boldsymbol{d}}(T_\lambda) = m < +\infty \text{ and } \mathsf{Im}[(T_\lambda)^m] \text{ is closed}\}.$$

The following lemma is the key to prove Theorem 3.8.

LEMMA 3.7. Let $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$, then

$$\mathsf{E}(T) = \{\lambda \in \sigma(T) \,:\, \widetilde{\boldsymbol{a}}(T_\lambda) < +\infty \;\text{ and }\; \widetilde{\boldsymbol{d}}(T_\lambda) < +\infty\}.$$

Proof. Assume that T has finite g-ascent and g-descent. First, by Lemma 2.2 and Lemma 2.4, we note that $\widetilde{\boldsymbol{a}}(T) = \widetilde{\boldsymbol{d}}(T) = p(T)$. Let $m = \widetilde{\boldsymbol{d}}(T)$, $S = T_{|\mathsf{Im}(T^m)}$ be the restriction of T to $\mathsf{Im}(T^m)$ and $\widetilde{T_m}$ be the relation induced by T on $\mathsf{H}/\mathsf{ker}(T^m)$, then by [9, Lemma 5.1],

$$\dim \mathsf{Im}(T^m) \cap \mathsf{ker}(T^k) = \alpha(S^k) \le k\,\alpha(S) = k\widetilde{\alpha}_m(T) = 0$$

and

$$\dim \mathsf{H}/[\mathsf{Im}(T^k) + \mathsf{ker}(T^m)] = \beta(\widetilde{T_m}^{k}) \le k\,\beta(\widetilde{T_m}) = k\,\widetilde{\beta}_m(T) = 0,$$

for all $k \in \mathbb{N}$. Therefore

$$\mathsf{Im}(T^m) \cap \mathsf{ker}(T^k) = \{0\} \;\text{ and }\; \mathsf{Im}(T^k) + \mathsf{ker}(T^m) = \mathsf{H}, \quad \forall\, k \ge 0. \tag{1}$$

Using now the equality (1) for $k = m$ and [7, Propositions 3.10 and 4.8, Lemma 4.2], we get $\mathsf{Im}(T^m)$ is closed. From (1) and Theorem 3.5, we deduce that there exists $\varepsilon > 0$ such that $\lambda I - T$ is bijective, for every $0 < |\lambda| < \varepsilon$. Consequently if $0 \in \sigma(T)$, then 0 is an isolated point of $\sigma(T)$. This completes the proof of Lemma 3.7. ∎

The following theorem is an extension of [2, Theorem 2.7] and [5, Theorem 3.10], to case of closed linear relations.

THEOREM 3.8. *Let $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$, then*

$$\varrho_{asc}^{e,g}(T) \cap \partial\sigma(T) = \varrho_{asc}^{g}(T) \cap \partial\sigma(T)$$
$$= \varrho_{asc}^{e}(T) \cap \partial\sigma(T) = \varrho_{asc}(T) \cap \partial\sigma(T) = \mathsf{E}(T).$$

Proof. By arguing as in the proof of [5, Theorem 3.10], with Theorem 3.4 and Lemma 3.7, we get the result. ∎

As an immediate consequence of Theorem 3.8, we have the following result.

COROLLARY 3.9. *Let $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$ such that $\varrho(T) \neq \emptyset$. Then the following assertions are equivalent :*

(*i*) $\sigma_{asc}(T) = \emptyset$;

(*ii*) $\sigma_{asc}^{e}(T) = \emptyset$;

(*iii*) $\partial\sigma(T) \subseteq \varrho_{asc}(T)$;

(*iv*) $\partial\sigma(T) \subseteq \varrho_{asc}^{e}(T)$;

(*v*) $\sigma(T) = \mathsf{E}(T)$.

Remark 3.10. We note that Corollary 3.9 in general is not true if $\varrho(T) = \emptyset$. Indeed, let $\mathsf{M} \neq \{0\}$ be a closed subspace of H and let $T : \mathsf{H} \longrightarrow \mathsf{H}$ be the linear relation defined by $Tx = \mathsf{M}$ for all $x \in \mathsf{H}$. It is clear that $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$. For each $\lambda \in \mathbb{C}$ and for each $n \in \mathbb{N}\backslash\{0\}$, we have

$$(T_\lambda)^n x = \lambda^n x + \mathsf{M}, \ \forall x \in \mathsf{H}.$$

Since

$$\mathsf{ker}[(T_\lambda)^n] = \begin{cases} \mathsf{H} & \text{if } \lambda = 0 \\ \mathsf{M} & \text{if } \lambda \neq 0 \end{cases} \neq \{0\} \quad \text{and} \quad \mathsf{Im}[(T_\lambda)^n] = \begin{cases} \mathsf{H} & \text{if } \lambda \neq 0 \\ \mathsf{M} & \text{if } \lambda = 0, \end{cases}$$

it follows that $\boldsymbol{a}(T_\lambda) = 1$ and $\mathsf{Im}(T_\lambda) + \mathsf{ker}(T_\lambda) = \mathsf{H}$, for all $\lambda \in \mathbb{C}$. This implies that $\varrho(T) = \emptyset$ and $\varrho_{asc}(T) = \mathbb{C}$. Hence, $\varrho_{asc}^{e}(T) = \mathbb{C}$ and T satisfies the conditions (*i*)-(*iv*) of Corollary 3.9. Since $\mathsf{E}(T)$ consists of isolated points of $\sigma(T)$, we deduce that $\mathsf{E}(T) = \emptyset$, and thus T does not satisfy the condition (*v*).

4. Spectral mapping theorems of g-ascent and essential g-ascent spectrums

We start this section with the following proposition.

Proposition 4.1. *Let $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$ be everywhere defined and suppose that $T^k(0)$ is closed for all $k \in \mathbb{N}$. If $n = \boldsymbol{a_e}(T)$ (resp. $\boldsymbol{a}(T)$, $\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T)$, $\widetilde{\boldsymbol{a}}(T)$) is finite, then $\mathsf{Im}(T) + \mathsf{ker}(T^n)$ is closed if and only if $\mathsf{Im}(T^{n+1})$ is closed.*

Proof. First, from [7, Lemma 4.6, Corollary 4.9], we get $T^k \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$, for all $k \in \mathbb{N}$. Put $n = \boldsymbol{a_e}(T)$ (resp. $\boldsymbol{a}(T)$, $\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T)$, $\widetilde{\boldsymbol{a}}(T)$). If $\mathsf{Im}(T) + \mathsf{ker}(T^n)$ is closed, by [6, Lemma 2.9] and Lemma 3.2, we get $\mathsf{Im}(T^{n+1}) + \mathsf{ker}(T^n)$ is closed and by [9, Lemma 3.2], we have

$$\begin{aligned}&\dim[\mathsf{Im}(T^{n+1}) \cap \mathsf{ker}(T^n)]/[T^{n+1}(0) \cap \mathsf{ker}(T^n)]\\ &= \dim \mathsf{ker}(T^{2n+1})/\mathsf{ker}(T^{n+1}) = \sum_{i=1}^{n} \alpha_{n+i}(T) \leq n\, \alpha_{n+1}(T) < +\infty.\end{aligned}$$

Then from [7, Propositions 3.9, 3.10, 4.8, Lemma 4.2], it follows that $\mathsf{Im}(T^{n+1})$ is closed. Now, suppose that $\mathsf{Im}(T^{n+1})$ is closed, from [3, Lemma 2.4], we obtain

$$\mathsf{Im}(T) + \mathsf{ker}(T^n) = T^{-n}[\mathsf{Im}(T^{n+1})]$$

is closed, which completes the proof. ∎

Corollary 4.2. *Let $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$ be everywhere defined and suppose that $T^n(0)$ is closed for all $n \in \mathbb{N}$. Then*

(i) $\varrho^{g}_{asc}(T) = \{\lambda \in \mathbb{C} : \widetilde{\boldsymbol{a}}(T_\lambda) < +\infty \text{ and } \mathsf{Im}[(T_\lambda)^{\widetilde{\boldsymbol{a}}(T_\lambda)+1}] \text{ is closed}\}$,

(ii) $\varrho^{e,g}_{asc}(T) = \{\lambda \in \mathbb{C} : \widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T_\lambda) < +\infty \text{ and } \mathsf{Im}[(T_\lambda)^{\widetilde{\boldsymbol{a}}_{e}(T_\lambda)+1}] \text{ is closed}\}$,

(iii) $\varrho_{asc}(T) = \{\lambda \in \mathbb{C} : \boldsymbol{a}(T_\lambda) < +\infty \text{ and } \mathsf{Im}[(T_\lambda)^{\boldsymbol{a}(T_\lambda)+1}] \text{ is closed}\}$,

(iv) $\varrho^{e}_{asc}(T) = \{\lambda \in \mathbb{C} : \boldsymbol{a_e}(T_\lambda) < +\infty \text{ and } \mathsf{Im}[(T_\lambda)^{\boldsymbol{a_e}(T_\lambda)+1}] \text{ is closed}\}$.

Let $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{H})$, for every non-constant polynomial $P = \prod_{i=1}^{n} (\lambda_i - X)^{\alpha_i}$, with coefficients in $\mathbb{C}$, we can associate the linear relation $P(T) \in \mathcal{L}_{\mathcal{R}}(\mathsf{H})$ defined by :

$$P(T) := \prod_{i=1}^{n} (\lambda_i I - T)^{\alpha_i}.$$

Remark 4.3. Let $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$ be everywhere defined such that $\varrho(T) \neq \emptyset$. If P is a non-constant complex polynomial, from Corollary 4.2 and [3, Theorem 4.7], it follows that

$$\sigma_{asc}(P(T)) = P(\sigma_{asc}(T)) \text{ and } \sigma^e_{asc}(P(T)) = P(\sigma^e_{asc}(T)).$$

For $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{H})$, we remark that

$$\boldsymbol{d}(T^{-1}) = \inf\{n \in \mathbb{N} : \mathcal{D}(T^n) = \mathcal{D}(T^{n+1})\},$$

where as usual the infimum over the empty set is taken to be $+\infty$. Hence, if $\boldsymbol{d}(T^{-1}) < +\infty$ then

$$\mathcal{D}(T^{\boldsymbol{d}(T^{-1})}) = \mathcal{D}(T^{\boldsymbol{d}(T^{-1})+n}) \subseteq \mathcal{D}(T^n), \quad \forall\ n \in \mathbb{N}.$$

EXAMPLE 4.4. Let H be a separable Hilbert space and let $K \in \mathcal{L}_{\mathcal{R}}(\mathsf{H})$. Consider the linear relation $T : \bigoplus_{i=0}^{\infty} \mathsf{H} \longrightarrow \bigoplus_{i=0}^{\infty} \mathsf{H}$ defined by $T(h_0 \oplus h_1 \oplus h_2 \oplus \ldots) = K(h_1) \oplus h_2 \oplus h_3 \oplus \ldots$. Clearly,

$$\mathsf{Im}(T^2) = \mathsf{Im}(T) \text{ and } \mathcal{D}(T^k) = \mathsf{H} \oplus \bigoplus_{i=1}^{i=k} \mathcal{D}(K) \oplus \bigoplus_{i=k+1}^{\infty} \mathsf{H},\ \forall k \geq 1.$$

Hence, if $\mathcal{D}(K) \subsetneq \mathsf{H}$ then $\boldsymbol{d}(T^{-1}) = +\infty$, and, if $\mathcal{D}(K) = \mathsf{H}$ then $\boldsymbol{d}(T^{-1}) = 0$. Let $S = T^{-1}$, so if $\mathsf{Im}(K) \subsetneq \mathsf{H}$ then $\boldsymbol{d}(S^{-1}) = \boldsymbol{d}(T) = 1$, and, $\boldsymbol{d}(S^{-1}) = 0$ when $\mathsf{Im}(K) = \mathsf{H}$.

Let us assume that T is a range space relation (see, Definition 3.1) such that $q = \boldsymbol{d}(T^{-1})$. It is clear that if $P = (\lambda_1 - X)^{\alpha_1}(\lambda_2 - X)^{\alpha_2} \cdots (\lambda_m - X)^{\alpha_m}$ is a complex polynomial then $P(T)$ is a range space relation (see, [7, Propositions 4.7, 4.8]) and $j = \boldsymbol{d}(P(T)^{-1}) \leq q$ according to [8, Theorem 3.2]. Furthermore, if P is a non-constant polynomial then $\mathcal{D}([P(T)]^j) = \mathcal{D}(T^q)$.

Let us assume that P is a non-constant complex polynomial and $T(0) \subseteq \mathcal{D}(T^q)$, then for all $n \geq q$ and $m \in \mathbb{N}$, we have $\mathsf{Im}\big([P(T)]^n\big) \subseteq \mathcal{D}(T^m)$. Indeed, we prove by induction that $T^n(0) \subseteq \mathcal{D}(T^q)$. The cases $n = 0$, 1 are obvious. Suppose that $T^n(0) \subseteq \mathcal{D}(T^q)$, then

$$\begin{aligned} T^{n+1}(0) = T\big(T^n(0)\big) \subseteq T\big(\mathcal{D}(T^{q+1})\big) &= TT^{-1}(\mathcal{D}(T^q)) \\ &= \mathcal{D}(T^q) \cap \mathsf{Im}(T) + T(0) \subseteq \mathcal{D}(T^q). \end{aligned}$$

This implies that $T^n(0) \subseteq \mathcal{D}(T^q)$ for all $n \in \mathbb{N}$, and consequently $T^n(0) \subseteq \mathcal{D}(T^m)$ for all n, $m \in \mathbb{N}$. Finally, by [10, Lemma 4.1], we get $\mathsf{Im}\big([P(T)]^n\big) \subseteq \mathcal{D}(T^q) \subseteq \mathcal{D}(T^m)$, for every $n \geq q$ and $m \in \mathbb{N}$.

In the following, we define

$$\begin{aligned}\Upsilon(\mathsf{H}) = \big\{&T \in \mathcal{L}_{\mathcal{R}}(\mathsf{H}) \,:\, T \text{ is a range space relation, } q = \boldsymbol{d}(T^{-1}) < +\infty,\\ &T(0) \subseteq \mathcal{D}(T^q),\ T^n(0),\ \mathcal{D}(T^{n+2}) \text{ and } \mathsf{Im}(T_\lambda) + \mathcal{D}(T^q) \text{ are closed,}\\ &\forall\ \lambda \in \mathbb{C},\ \forall n \in \mathbb{N}\big\}.\end{aligned}$$

Clearly, $\Upsilon(\mathsf{H}) \neq \emptyset$, because $T \in \Upsilon(\mathsf{H})$, when T is a closed linear relation everywhere defined such that $T^n(0)$ is closed for all $n \in \mathbb{N}$.

For family of vectors $(x_i)_{i\in I}$ in H, we denote by $\mathsf{Vect}(x_i,\, i \in I)$, the vector subspace generated by $(x_i)_{i\in I}$.

Example 4.5.

(*i*) Let H be a separable Hilbert space and let $K \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$ such that $\mathcal{D}(K) \subsetneq \mathsf{H}$ is closed. Let $\mathcal{H} = \bigoplus_{i=0}^{3} \mathsf{H}$ and consider the linear relation $T : \mathcal{H} \longrightarrow \mathcal{H}$ defined by $T(h_0 \oplus h_1 \oplus h_2 \oplus h_3) = K(h_1) \oplus h_2 \oplus h_3 \oplus h_3$. Clearly, $T^n(0 \oplus 0 \oplus 0 \oplus 0) = K(0) \oplus 0 \oplus 0 \oplus 0$ is closed for all $n \geq 1$ and

$$\mathcal{D}(T^k) = \begin{cases} \mathsf{H} \oplus \mathcal{D}(K) \oplus \mathsf{H} \oplus \mathsf{H} & \text{if } k = 1\\ \mathsf{H} \oplus \mathcal{D}(K) \oplus \mathcal{D}(K) \oplus \mathsf{H} & \text{if } k = 2\\ \mathsf{H} \oplus \mathcal{D}(K) \oplus \mathcal{D}(K) \oplus \mathcal{D}(K) & \text{if } k \geq 3\end{cases}$$

is closed. Hence $\boldsymbol{d}(T^{-1}) = 3$ and $T(0 \oplus 0 \oplus 0 \oplus 0) \subseteq \mathcal{D}(T^3)$. It is not difficult to see that

$$\mathsf{Im}(T_\lambda) + \mathcal{D}(T^3) = \begin{cases} \mathsf{H} \oplus \mathsf{H} \oplus \mathsf{H} \oplus \mathcal{D}(K) & \text{if } \lambda = 1\\ \mathsf{H} \oplus \mathsf{H} \oplus \mathsf{H} \oplus \mathsf{H} & \text{if } \lambda \neq 1\end{cases}$$

is closed. Since $T \in \mathcal{C}_{\mathcal{R}}(\mathcal{H})$, it follows that $T \in \Upsilon(\mathcal{H})$. Assume that $K(0) \subsetneq \mathsf{H}$ and let $S = T^{-1}$. It is easy to see that

$$T^n(h_0 \oplus h_1 \oplus h_2 \oplus h_3) = K(h_3) \oplus h_3 \oplus h_3 \oplus h_3,\ \forall h_0 \in \mathsf{H},\quad \forall\ h_1, h_2, h_3 \in \mathcal{D}(K),$$

for all $n \geq 3$. Therefore $\boldsymbol{d}(S^{-1}) = \boldsymbol{d}(T) \leq 3$. Let $h \in \mathsf{H}\backslash K(0)$ and $\xi = h \oplus 0 \oplus 0 \oplus 0$, then $\xi \in S(0) = \mathsf{ker}(T) = \mathsf{H} \oplus \mathsf{ker}(K) \oplus \{0\} \oplus \{0\}$ and $\xi \notin \mathcal{D}(S^3) = \mathsf{Im}(T^3)$. Consequently, $S \notin \Upsilon(\mathcal{H})$.

(*ii*) Let H be a separable Hilbert space and $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H. Define the following operators T and L in H by

$$\mathcal{D}(T) = \mathcal{D}(L) = \mathsf{Vect}(e_n : n \geq 2),\ T(e_n) = e_{n+1}$$

and $L(e_n) = e_{n-1}$, for all $n \geq 2$. It is clear that $\mathcal{D}(T^k) = \mathcal{D}(T)$ and $\mathcal{D}(L^k) = \mathsf{Vect}(e_n : n \geq 1 + k)$, for all $k \geq 1$ and hence $\boldsymbol{d}(T^{-1}) = 1$ and $\boldsymbol{d}(L^{-1}) = +\infty$ ($L \notin \Upsilon(\mathsf{H})$). Since $T \in \mathcal{C_R}(\mathsf{H})$, $\mathsf{Im}(T_\lambda) \subseteq \mathcal{D}(T)$ for all $\lambda \in \mathbb{C}$ and $\mathcal{D}(T)$ is closed, then $T \in \Upsilon(\mathsf{H})$.

(*iii*) Let T be defined as in (*ii*) and $k \geq 2$. Now, we define the following relation $S := \mathsf{Im}(T^k) + T$ (i.e., $S(x) = \{y + z : y \in \mathsf{Im}(T^k),\ z \in T(x)\}$, for all $x \in \mathcal{D}(T)$). Since $S(0) = \mathsf{Im}(T^k)$ is closed (because $\beta(T) = 3$), T is a closed operator and $\mathcal{D}(S) = \mathcal{D}(T)$ is closed, then

$$\|Q_S S(x)\| = \|\overline{Tx}\| \leq \|Tx\| \leq \|T\|\,\|x\|, \text{ for all } x \in D(S).$$

This proves that $Q_S S$ is closed and by [4, Proposition II.5.3], we get that $S \in \mathcal{C_R}(\mathsf{H})$. It is clear that

$$S^j = \begin{cases} \mathsf{Im}(T^k) + T^j & \text{if } j < k \\ \mathsf{Im}(T^k) & \text{if } j \geq k. \end{cases}$$

For all $j \geq 1$, we have $\mathcal{D}(S^j) = \mathcal{D}(T)$ and $S^j(0) = \mathsf{Im}(T^k)$ are closed in H, and from this we get that $\boldsymbol{d}(S^{-1}) = 1$ and $S(0) \subseteq \mathcal{D}(S)$. Moreover, for all $\lambda \in \mathbb{C}$, we see that

$$\mathsf{Im}(S_\lambda) + \mathcal{D}(S) \;=\; \mathsf{Im}(T_\lambda) + \mathsf{Im}(T^k) + \mathcal{D}(T) = \mathcal{D}(T)$$

is closed. Now, we can conclude that $S \in \Upsilon(\mathsf{H})$.

(*iv*) Let T be defined as in Example 2.1, where M is a closed subspace of H and $\mathsf{M} \subsetneq \mathsf{H}$. It is easy to see that $\mathsf{G}(T) = \mathsf{M} \times \mathsf{M}$ and

$$T^n(0) = \mathcal{D}(T^n) = \mathsf{Im}(T_\lambda) = \mathsf{M},\ \ \forall\, n \geq 1,\ \lambda \in \mathbb{C}.$$

From this it follows that $T \in \Upsilon(\mathsf{H})$.

(*v*) Let L be defined as in Example 2.1. It is not difficult to see that L is a closed relation,

$$\boldsymbol{d}(L^{-1}) = 1,\ \mathcal{D}(L^n) = \mathsf{N} \text{ is closed},\ \mathcal{D}(L^n) + \mathsf{Im}(L_\lambda) = \mathsf{H},\ \forall n \geq 1,\ \lambda \in \mathbb{C}.$$

But $L(0) = \mathsf{M} \not\subset \mathcal{D}(L)$, so that $L \notin \Upsilon(\mathsf{H})$.

Let us show that if $T \in \Upsilon(\mathsf{H})$ and $P = (\lambda_1 - X)^{\alpha_1}(\lambda_2 - X)^{\alpha_2} \cdots (\lambda_m - X)^{\alpha_m}$ is a non-constant complex polynomial, then $A = P(T) \in \Upsilon(\mathsf{H})$. Put $q = \boldsymbol{d}(T^{-1})$ and define the following relation

$$\begin{array}{rccc} \overline{T}\ : & \mathcal{D}(\overline{T}) \subseteq \mathsf{H}/\mathcal{D}(T^q) & \longrightarrow & \mathsf{H}/\mathcal{D}(T^q) \\ & \overline{x} & \longmapsto & \overline{Tx}. \end{array}$$

Note that the linear relation $\overline{T}$ is well-defined. Indeed, since $T(0) \subseteq \mathcal{D}(T^q)$,

$$\begin{aligned} T(\mathcal{D}(T^q)) &= T(\mathcal{D}(T^{q+1})) = TT^{-1}(\mathcal{D}(T^q)) \\ &= \mathcal{D}(T^q) \cap \mathsf{Im}(T) + T(0) \subseteq \mathcal{D}(T^q), \end{aligned}$$

which implies that if $x_1,\, x_2 \in \mathcal{D}(T)$ such that $\overline{x_1} = \overline{x_2}$, then $\overline{Tx_1} = \overline{Tx_2}$. We remark also that $\overline{T}(\overline{0}) = \overline{T(0)} = \overline{0}$ and thus $\overline{T}$ is an unbounded operator. Now, let $\lambda \in \mathbb{C}$ and $\overline{x} \in \ker(\lambda I - \overline{T})$, then $\overline{T_\lambda x} = \overline{T_\lambda(0)} = \overline{0}$, which implies that $T_\lambda x \subseteq \mathcal{D}(T^q)$ and $x \in \mathcal{D}(T^{q+1}) = \mathcal{D}(T^q)$. Consequently, $\overline{x} = \overline{0}$, and thus $\ker(\lambda I - \overline{T}) = \{\overline{0}\}$.

On the other hand, it is clear that $\mathsf{Im}(\lambda I - \overline{T}) = \big(\mathsf{Im}(T_\lambda) + \mathcal{D}(T^q)\big)/\mathcal{D}(T^q)$ is closed. As in the proof of Lemma 3.3, we obtain $\lambda I - \overline{T}$ is a range space operator. Now, applying [7, Lemma 4.6], we get $\lambda I - \overline{T}$ is a closed operator, and thus $\lambda I - \overline{T} \in \Phi_+(\mathsf{H}/\mathcal{D}(T^q))$. Recall that, if $S,\, L \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$ are two operators such that $L \in \Phi_+(\mathsf{H})$ and $\mathsf{Im}(S)$ is closed, then $LS \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$ and $\mathsf{Im}(LS)$ is closed. Since, for every $i,\, j \in \{1, 2, \cdots, m\}$, $\lambda_i I - \overline{T} \in \Phi_+(\mathsf{H}/\mathcal{D}(T^q))$ and $\mathsf{Im}(\lambda_j I - \overline{T})$ is closed, we deduce that $(\lambda_i I - \overline{T})(\lambda_j I - \overline{T}) \in \mathcal{C}_{\mathcal{R}}(\mathsf{H}/\mathcal{D}(T^q))$ and $\mathsf{Im}[(\lambda_i I - \overline{T})(\lambda_j I - \overline{T})]$ is closed. Consequently, from $\ker\big((\lambda_i I - \overline{T})(\lambda_j I - \overline{T})\big) = \{\overline{0}\}$, it follows that $(\lambda_i I - \overline{T})(\lambda_j I - \overline{T}) \in \Phi_+(\mathsf{H}/\mathcal{D}(T^q))$. Therefore $\mathsf{Im}(P(\overline{T})) = \big(\mathsf{Im}(P(T)) + \mathcal{D}(T^q)\big)/\mathcal{D}(T^q)$ is closed, and finally we obtain $\mathsf{Im}(A) + \mathcal{D}(A^{\boldsymbol{d}(A^{-1})}) = \mathsf{Im}\big(P(T)\big) + \mathcal{D}(T^q)$ is closed.

Now, let $\lambda \in \mathbb{C}$ and put $Q = \lambda - P = a \prod\limits_{i=1}^{m} (\mu_i - X)^{\beta_i}$. Arguing in the same way as previous, we can conclude that $\mathsf{Im}(A_\lambda) + \mathcal{D}(A^{\boldsymbol{d}(A^{-1})})$ is closed. Let $k = \sum\limits_{i=0}^{m} \alpha_i$ and $n \in \mathbb{N}$, by [8, Theorems 3.2, 3.6], we get $A^n(0) = T^{kn}(0)$ and $\mathcal{D}(A^{n+2}) = \mathcal{D}(T^{(n+2)k})$ are closed, and $A(0) \subseteq \mathcal{D}(T^q) = \mathcal{D}(A^{\boldsymbol{d}(A^{-1})})$. This proves that, $A \in \Upsilon(\mathsf{H})$.

The aim of this section is to establish the spectral mapping theorem of g-ascent and essential g-ascent spectrums of a closed linear relation $T \in \Upsilon(\mathsf{H})$. First, we have the following remark.

Remark 4.6. (i) If $T \in \Upsilon(\mathsf{H})$, then $P(T) \in \mathcal{C_R}(\mathsf{H})$, for any complex polynomial P of degree $n \geq \min\{\boldsymbol{d}(T^{-1}), 2\}$. Indeed, we have $\mathcal{D}[P(T)] = \mathcal{D}(T^n)$ and $P(T)(0) = T^n(0)$ are closed, and $P(T)$ is a range space relation. So, by [7, Lemma 4.6], it follows that $P(T) \in \mathcal{C_R}(\mathsf{H})$.

(ii) Let $T \in \Upsilon(\mathsf{H})$ such that $\mathcal{D}(T)$ is closed or T be a closed relation. By (i) and [7, Lemma 4.6], we deduce that $P(T) \in \mathcal{C_R}(\mathsf{H})$, for any complex polynomial P.

The next lemma is used to prove Lemma 4.8.

LEMMA 4.7. *Let* $T \in \mathcal{L_R}(\mathsf{H})$, $P = \sum\limits_{i=0}^{m} a_i X^i = \alpha \prod\limits_{i=1}^{m} (\lambda_i - X)$ *and* $Q = \sum\limits_{i=0}^{n} b_i X^i = \beta \prod\limits_{i=1}^{n} (\lambda_i - X)$ *be non-constants complex polynomials. Then*

(i) $P(T) = \sum\limits_{i=0}^{m} a_i T^i$,

(ii) $P(T) + Q(T) = (P+Q)(T) + T^s - T^s$, *where* $s = \max\{n, m\}$.

Proof. It is easy to see that if $\xi \in \mathbb{C}$ and $i, j \in \mathbb{N}$ such that $j \leq i$, then

$$\xi T^j(x) + T^i(0) = T^j(\xi x) + T^i(0), \ \ \forall\, x \in \mathcal{D}(T^j). \tag{1}$$

It follows from this that for all $\mu \in \mathbb{C}\backslash\{0\}$,

$$\xi T^j(x) + \mu T^i(x) = T^j(\xi x) + \mu T^i(x), \ \ \forall\, x \in \mathcal{D}(T^i). \tag{2}$$

(i) We will prove that $P(T) = \sum\limits_{i=0}^{m} a_i T^i$ and $a_m = (-1)^m \alpha$. By [4, Proposition I.4.2], we know that if R, S, $L \in \mathcal{L_R}(\mathsf{H})$ such that $\mathcal{D}[L(R+S)] = \mathcal{D}(LR+LS)$, then $L(R+S) = LR + LS$. So by [8, Theorem 3.2] and (2), we get

$$\begin{aligned}(\lambda_1 I - T)(\lambda_2 I - T)x &= (\lambda_1 I - T)(\lambda_2 x) - (\lambda_1 I - T)Tx \\ &= \lambda_1\lambda_2 x - T(\lambda_2 x) - T(\lambda_1 x) + T^2(x) \\ &= \lambda_1\lambda_2 x - T[(\lambda_2 + \lambda_1)x] + T^2(x) \\ &= \lambda_1\lambda_2 x - (\lambda_2 + \lambda_1)T(x) + T^2(x), \ \ \forall\, x \in \mathcal{D}(T^2).\end{aligned}$$

Suppose that

$$\alpha(\lambda_1 I - T)(\lambda_2 I - T)\cdots(\lambda_{m-1} I - T) = \sum_{i=0}^{m-2} \alpha_i T^i + \alpha\,(-1)^{m-1} T^{m-1}$$

and let us show that

$$\alpha(\lambda_1 I - T)(\lambda_2 I - T)\cdots(\lambda_m I - T) = \sum_{i=0}^{m-1} \gamma_i T^i + \alpha\,(-1)^m T^m.$$

By [8, Corollary 2.1] and (2), we obtain

$$\begin{aligned}
P(T)x &= \alpha(\lambda_1 I - T)(\lambda_2 I - T)\cdots(\lambda_m I - T)x \\
&= (\lambda_m I - T)[\alpha(\lambda_1 I - T)(\lambda_2 I - T)\cdots(\lambda_{m-1} I - T)]x \\
&= (\lambda_m I - T)\Big(\sum_{i=0}^{m-2} \alpha_i T^i + \alpha\,(-1)^{m-1}T^{m-1}\Big)x \\
&= (\lambda_m I - T)\Big(\sum_{i=0}^{m-2} T^i(\alpha_i x) + \alpha\,(-1)^{m-1}T^{m-1}x\Big) \\
&= \sum_{i=0}^{m-2} T^i(\lambda_m I - T)(\alpha_i x) + \alpha\,(-1)^{m-1}T^{m-1}(\lambda_m I - T)x \\
&= \sum_{i=0}^{m-2} T^i(\lambda_m \alpha_i x) - \sum_{i=0}^{m-2} T^{i+1}(\alpha_i x) \\
&\quad + \alpha\,(-1)^{m-1}T^{m-1}(\lambda_m x) + \alpha\,(-1)^m T^m x \\
&= \sum_{i=0}^{m-2} T^i(\lambda_m \alpha_i x) - \sum_{i=1}^{m-1} T^i(\alpha_{i-1} x) \\
&\quad + \alpha\,(-1)^{m-1}T^{m-1}(\lambda_m x) + \alpha\,(-1)^m T^m x
\end{aligned}$$

So,

$$\begin{aligned}
P(T)x &= \lambda_m \alpha_0 x + \sum_{i=1}^{m-2} T^i[(\lambda_m \alpha_i - \alpha_{i-1})x] \\
&\quad + T^{m-1}[(\alpha\,(-1)^{m-1}\lambda_m - \alpha_{m-2})x] + \alpha\,(-1)^m T^m x \\
&= \underbrace{\lambda_m \alpha_0}_{\gamma_0} x + \sum_{i=1}^{m-2} \underbrace{(\lambda_m \alpha_i - \alpha_{i-1})}_{\gamma_i} T^i(x) \\
&\quad + \underbrace{(\alpha\,(-1)^{m-1}\lambda_m - \alpha_{m-2})}_{\gamma_{m-1}} T^{m-1}(x) + \alpha\,(-1)^m T^m x \\
&= \sum_{i=0}^{m-1} \gamma_i T^i x + \alpha\,(-1)^m T^m x, \ \ \forall\, x \in \mathcal{D}(T^m).
\end{aligned}$$

This shows that $P = \sum_{i=0}^{m-1} \gamma_i X^i + \alpha\,(-1)^m X^m$ and hence $a_i = \gamma_i$, for all $i = 1, \cdots, m-1$ and $a_m = \alpha\,(-1)^m$.

(ii) Assume that $m \leq n$, from (i) and (1), we see that

$$\begin{aligned}
[P(T)+Q(T)](x) &= [P(T)+Q(T)](x) + [P(T)+Q(T)](0) \\
&= \sum_{i=0}^{m} a_i T^i x + \sum_{i=0}^{n} b_i T^i x + T^n(0) \\
&= \sum_{i=0}^{m} (a_i T^i x + b_i T^i x) + \sum_{i=m+1}^{n} b_i T^i x + T^n(0) \\
&= \sum_{i=0}^{m} [T^i(a_i x) + T^i(b_i x)] + \sum_{i=m+1}^{n} b_i T^i(x) + T^n(0) \\
&= \sum_{i=0}^{m} T^i[(a_i + b_i)x] + \sum_{i=m+1}^{n} b_i T^i(x) + T^n(0) \\
&= \sum_{i=0}^{m} (a_i + b_i) T^i(x) + \sum_{i=m+1}^{n} b_i T^i(x) + T^n(0) \\
&= \sum_{i=0}^{n} \omega_i T^i(x) + T^n(0), \quad \forall\, x \in \mathcal{D}(T^n),
\end{aligned}$$

Since $P + Q = \sum_{i=0}^{n} \omega_i X^i$, then

$$P(T) + Q(T) = (P+Q)(T) + T^n - T^n.$$

This completes the proof. ∎

The following result is an improvement of [5, Lemma 4.4] to closed linear relations.

LEMMA 4.8. *Let $T \in \mathcal{L_R}(\mathsf{H})$ and let P and Q be two relatively prime complex polynomials. If $A = P(T)$ and $B = Q(T)$, then*

(i) $\mathsf{Im}(A^n B^n) = \mathsf{Im}(A^n) \cap \mathsf{Im}(B^n)$, *for all $n \in \mathbb{N}$,*

(ii) $\mathsf{ker}(A^n B^n) = \mathsf{ker}(A^n) + \mathsf{ker}(B^n)$, *for all $n \in \mathbb{N}$,*

(iii) $\mathsf{ker}(A^n) \subseteq \mathsf{Im}(B^m)$ *and* $\mathsf{ker}(B^n) \subseteq \mathsf{Im}(A^m)$, *for all n, $m \in \mathbb{N}$,*

(iv) $\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(AB) = \max\{\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(A), \widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(B)\}$ *and* $\widetilde{\boldsymbol{a}}(AB) = \max\{\widetilde{\boldsymbol{a}}(A), \widetilde{\boldsymbol{a}}(B)\}$.

In addition, assume that $T \in \Upsilon(\mathsf{H})$, (*v*) *if* $\max\{\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(A),\ \widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(B)\} < +\infty$, *then* $\mathsf{Im}(A) + \mathsf{ker}(A^{\widetilde{\boldsymbol{a}}_e(A)})$ *and* $\mathsf{Im}(B) + \mathsf{ker}(B^{\widetilde{\boldsymbol{a}}_e(B)})$ *are both closed if and only if* $\mathsf{Im}(AB) + \mathsf{ker}[(AB)^{\widetilde{\boldsymbol{a}}_e(AB)}]$ *is closed,*

(*vi*) *if* $\max\{\widetilde{\boldsymbol{a}}(A),\ \widetilde{\boldsymbol{a}}(B)\} < +\infty$, *then* $\mathsf{Im}(A) + \mathsf{ker}(A^{\widetilde{\boldsymbol{a}}(A)})$ *and* $\mathsf{Im}(B) + \mathsf{ker}(B^{\widetilde{\boldsymbol{a}}(B)})$ *are both closed if and only if* $\mathsf{Im}(AB) + \mathsf{ker}[(AB)^{\widetilde{\boldsymbol{a}}(AB)}]$ *is closed.*

Proof. The proof is trivial when P or Q is a constant polynomial. Assume that P and Q are non-constants polynomials. So, $P = (\lambda_1 - X)^{\alpha_1}(\lambda_2 - X)^{\alpha_2} \cdots (\lambda_m - X)^{\alpha_m}$ and $Q = (\mu_1 - X)^{\beta_1}(\mu_2 - X)^{\beta_2} \cdots (\mu_s - X)^{\beta_s}$. First, it is clear that the assertions (*i*) and (*ii*) follow immediately from [8, Theorems 3.3, 3.4].

(*iii*) By [10, Lemma 7.2], we know that $\mathsf{ker}[(\lambda_i I - T)^{\alpha_i}] \subseteq \mathsf{Im}[(\mu_j I - T)^{\beta_j}]$, so from [8, Theorems 3.3, 3.4], $\mathsf{ker}(A^n) \subseteq \mathsf{Im}(B^m)$, for all $n,\ m \in \mathbb{N}$.

(*iv*) Let $n \in \mathbb{N}$, we have

$$\begin{aligned}\mathsf{Im}(A^nB^n) \cap \mathsf{ker}(AB) &= \mathsf{Im}(A^n) \cap \mathsf{Im}(B^n) \cap [\mathsf{ker}(A) + \mathsf{ker}(B)]\\ &= \mathsf{Im}(A^n) \cap [\mathsf{ker}(A) + \mathsf{Im}(B^n) \cap \mathsf{ker}(B)]\\ &= [\mathsf{Im}(A^n) \cap \mathsf{ker}(A)] + [\mathsf{Im}(B^n) \cap \mathsf{ker}(B)].\end{aligned}$$

Therefore

$$\widetilde{\alpha}_n(AB) = 0 \Longleftrightarrow \max\{\widetilde{\alpha}_n(B),\, \widetilde{\alpha}_n(A)\} = 0,$$

$$\widetilde{\alpha}_n(AB) < +\infty \Longleftrightarrow \max\{\widetilde{\alpha}_n(B),\, \widetilde{\alpha}_n(A)\} < +\infty.$$

(*v*) Let $n \in \mathbb{N}\backslash\{0\}$. Since P^n and Q^n are relatively prime, we know that there exist two polynomials $P_n = a\prod_{i=1}^{p}(\nu_i - X)^{r_i}$ and $Q_n = b\prod_{i=1}^{r}(\omega_i - X)^{j_i}$ such that $P^nP_n + Q^nQ_n = 1$. Let p_n (resp. k) be the degree of P_n (resp. P) and $\alpha(n) = nk + p_n$. Then, the degree of P^nP_n (resp. Q^nQ_n) is $\alpha(n)$ and by [8, Theorem 3.2], we get

$$\mathcal{D}[A^nP_n(T) + B^nQ_n(T)] = \mathcal{D}[A^nP_n(T)] \cap \mathcal{D}[B^nQ_n(T)] = \mathcal{D}(T^{\alpha(n)}).$$

Now, by Lemma 4.7, we obtain

$$A^nP_n(T)x + B^nQ_n(T)x = x + T^{\alpha(n)}(0), \quad \forall\, x \in \mathcal{D}(T^{\alpha(n)}). \tag{1}$$

If $n \geq q = \boldsymbol{d}(T^{-1})$, then $\alpha(n) \geq q$, and by (1), it is clear that

$$\mathcal{D}(T^q) = \mathcal{D}(T^{\alpha(n)}) \subseteq \mathsf{Im}(A^n) + \mathsf{Im}(B^n).$$

Since $\mathsf{Im}(A^n) \subseteq \mathcal{D}(T^q)$ and $\mathsf{Im}(B^n) \subseteq \mathcal{D}(T^q)$, for every $j \in \mathbb{N}$, it follows that

$$\mathcal{D}(T^q) = \mathsf{Im}(A^n) + \mathsf{Im}(B^n) = [\mathsf{ker}(A^j) + \mathsf{Im}(A^n)] + [\mathsf{Im}(B^n) + \mathsf{ker}(B^j)] \quad (2)$$

and

$$\begin{aligned} \mathsf{ker}(A^jB^j) + \mathsf{Im}(A^nB^n) &= \mathsf{ker}(A^j) + \mathsf{ker}(B^j) + \mathsf{Im}(A^n) \cap \mathsf{Im}(B^n) \\ &= [\mathsf{ker}(A^j) + \mathsf{Im}(A^n)] \cap \mathsf{Im}(B^n) + \mathsf{ker}(B^j) \quad (3) \\ &= [\mathsf{ker}(A^j) + \mathsf{Im}(A^n)] \cap [\mathsf{Im}(B^n) + \mathsf{ker}(B^j)]. \end{aligned}$$

From [7, Propositions 3.9, 3.10, 4.8, Lemma 4.2] and by using (2) and (3), we get

$$\mathsf{ker}(A^jB^j) + \mathsf{Im}(A^nB^n) \text{ is closed} \Leftrightarrow \begin{cases} \mathsf{ker}(A^j) + \mathsf{Im}(A^n), \\ \mathsf{ker}(B^j) + \mathsf{Im}(B^n) \end{cases} \text{are both closed.} \quad (4)$$

Let $n > q = \boldsymbol{d}(T^{-1})$ and

$$j = \widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(A) + \widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(B) + \widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(AB) + p(A) + p(B) + p(AB).$$

Let us assume that $\mathsf{Im}(AB) + \mathsf{ker}(A^jB^j)$ is closed. Since the degree of PQ is greater than or equal to two, then from Remark 4.6, $AB \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$. It follows from [6, Lemma 2.9] and Lemma 3.2 that $\mathsf{Im}(A^nB^n) + \mathsf{ker}(A^jB^j)$ is closed. Let us assume that $\mathsf{N} = \mathsf{Im}(A^nB^n) + \mathsf{ker}(A^jB^j)$ is closed. As $\mathcal{D}(B^{n-1}A^{n-1}) = \mathcal{D}(T^q)$ is closed and $\mathsf{Im}(B^{n-1}A^{n-1}) \subseteq \mathcal{D}(T^q)$,

$$\begin{array}{rcccl} S & : & \mathcal{D}(T^q) & \longrightarrow & \mathcal{D}(T^q) \\ & & x & \longmapsto & B^{n-1}A^{n-1}x \end{array}$$

is well-defined. From [7, Lemmas 4.6, 4.10, Proposition 4.8], it follows that $S \in \mathcal{C}_{\mathcal{R}}(\mathcal{D}(T^q))$. Because $\mathcal{D}(S) = \mathcal{D}(T^q)$ is closed and $S(0) \subseteq \mathsf{N} \subseteq \mathcal{D}(T^q)$, by [3, Lemma 2.4],

$$\big(\mathsf{Im}(AB) + \mathsf{ker}[(AB)^{j+n-1}]\big) \cap \mathcal{D}(T^q) = S^{-1}(\mathsf{N}) \text{ is closed.}$$

Since

$$(\mathsf{Im}(AB) + \mathsf{ker}[(AB)^{j+n-1}]) + \mathcal{D}(T^q) = \mathsf{Im}(AB) + \mathcal{D}[(AB)^{\boldsymbol{d}(A^{-1}B^{-1})}]$$

is closed, from [7, Propositions 3.9, 3.10, 4.8, Lemma 4.2], it follows that $\mathsf{Im}(AB) + \mathsf{ker}[(AB)^{j+n-1}]$ is closed. Now, by [6, Lemma 2.9], $\mathsf{Im}(AB) + \mathsf{ker}[(AB)^j]$ is closed, and thus

$$\mathsf{Im}(AB) + \mathsf{ker}(A^jB^j) \text{ is closed} \iff \mathsf{Im}(A^nB^n) + \mathsf{ker}(A^jB^j) \text{ is closed.} \quad (5)$$

Arguing in the same way as previous, we can conclude that

$$\begin{array}{lcl} \mathsf{Im}(A) + \mathsf{ker}(A^j) \text{ is closed} & \Longleftrightarrow & \mathsf{Im}(A^n) + \mathsf{ker}(A^j) \text{ is closed,} \\ \mathsf{Im}(B) + \mathsf{ker}(B^j) \text{ is closed} & \Longleftrightarrow & \mathsf{Im}(B^n) + \mathsf{ker}(B^j) \text{ is closed.} \end{array} \tag{6}$$

Finally, it follows from (4), (5), (6) and [6, Lemma 2.9] that $\mathsf{Im}(A)+\mathsf{ker}(A^{\widetilde{\boldsymbol{a}}_e(A)})$ and $\mathsf{Im}(B)+\mathsf{ker}(B^{\widetilde{\boldsymbol{a}}_e(B)})$ are both closed if and only if $\mathsf{Im}(AB)+\mathsf{ker}[(AB)^{\widetilde{\boldsymbol{a}}_e(AB)}]$ is closed.

(vi) Finally, by [6, Lemma 2.9] and the assertion (v), we see that the following assertions are equivalent :

$a)$ $\mathsf{Im}(A) + \mathsf{ker}(A^{\widetilde{\boldsymbol{a}}(A)})$ is closed and $\mathsf{Im}(B) + \mathsf{ker}(B^{\widetilde{\boldsymbol{a}}(B)})$ is closed,

$b)$ $\mathsf{Im}(A) + \mathsf{ker}(A^{\widetilde{\boldsymbol{a}}_e(A)})$ is closed and $\mathsf{Im}(B) + \mathsf{ker}(B^{\widetilde{\boldsymbol{a}}_e(B)})$ is closed,

$c)$ $\mathsf{Im}(AB) + \mathsf{ker}[(AB)^{\widetilde{\boldsymbol{a}}_e(AB)}]$ is closed,

$d)$ $\mathsf{Im}(AB) + \mathsf{ker}[(AB)^{\widetilde{\boldsymbol{a}}(AB)}]$ is closed.

The proof is complete. ∎

Theorem 4.9. *Let $T \in \Upsilon(\mathsf{H})$ be a closed linear relation. If A and B are defined as in* Lemma 4.8, then

$$0 \in \varrho^{e,g}_{asc}(AB) \Longleftrightarrow 0 \in \varrho^{e,g}_{asc}(A) \cap \varrho^{e,g}_{asc}(B)$$

and

$$0 \in \varrho^{g}_{asc}(AB) \Longleftrightarrow 0 \in \varrho^{g}_{asc}(A) \cap \varrho^{g}_{asc}(B).$$

Proof. This is an obvious consequence of Remark 4.6 and Lemma 4.8. ∎

Theorem 4.10. *Let $T \in \Upsilon(\mathsf{H})$ be a closed linear relation and $m \in \mathbb{N}\backslash\{0\}$. Then*

$$0 \in \varrho^{e,g}_{asc}(T) \Longleftrightarrow 0 \in \varrho^{e,g}_{asc}(T^m)$$

and

$$0 \in \varrho^{g}_{asc}(T) \Longleftrightarrow 0 \in \varrho^{g}_{asc}(T^m).$$

Proof. First, since $T \in \Upsilon(\mathsf{H})$ is closed, we obtain T^m is closed (see Remark 4.6). Let n, $m \in \mathbb{N}\backslash\{0\}$ and $S = T_{|\mathsf{Im}(T^{mn})}$ be the restriction of T to $\mathsf{Im}(T^{mn})$. From [9, Lemma 5.1], it follows that,

$$\widetilde{\alpha}_{nm}(T) \leq \widetilde{\alpha}_n(T^m) = \alpha(S^m) \leq m\alpha(S) = m\widetilde{\alpha}_{nm}(T),$$

and this proves that $\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T) < +\infty$ if and only if $\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T^m) < +\infty$. Now, put $k = \max\{p(T^m),\, p(T)\} < +\infty$, then by Lemma 2.2, $k \geq \max\{\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T),\, \widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T^m)\}$. Let $n > \boldsymbol{d}(T^{-1})$, as in the proof of Lemma 4.8 and according to [6, Lemma 2.9] and Lemma 3.2, we deduce that

$$\begin{aligned} \mathsf{Im}(T^m) + \mathsf{ker}(T^{m\,k}) \text{ is closed} \quad &\Longrightarrow \quad \mathsf{Im}(T^{m\,n}) + \mathsf{ker}(T^{m\,k}) \text{ is closed,} \\ &\Longrightarrow \quad \mathsf{Im}(T) + \mathsf{ker}(T^{m\,k}) \text{ is closed.} \end{aligned}$$

Hence, from [6, Lemma 2.9] and Lemma 3.2,

$$\begin{aligned} \mathsf{Im}(T) + \mathsf{ker}(T^{\widetilde{\boldsymbol{a}}_e(T)}) \text{ is closed} \quad &\Longleftrightarrow \quad \mathsf{Im}(T) + \mathsf{ker}(T^{m\,k}) \text{ is closed,} \\ &\Longleftrightarrow \quad \mathsf{Im}(T^m) + \mathsf{ker}(T^{m\,k}) \text{ is closed,} \\ &\Longleftrightarrow \quad \mathsf{Im}(T^m) + \mathsf{ker}[(T^m)^{\widetilde{\boldsymbol{a}}_e(T^m)}] \text{ is closed.} \end{aligned}$$

This gives

$$0 \in \varrho_{asc}^{e,\,g}(T) \Longleftrightarrow 0 \in \varrho_{asc}^{e,\,g}(T^m).$$

Furthermore, for every $m \in \mathbb{N}\backslash\{0\}$, $\widetilde{\boldsymbol{a}}(T) < +\infty$ if and only if $\widetilde{\boldsymbol{a}}(T^m) < +\infty$. So that

$$\begin{aligned} \mathsf{Im}(T) + \mathsf{ker}(T^{\widetilde{\boldsymbol{a}}(T)}) \text{ is closed} \quad &\Longleftrightarrow \quad \mathsf{Im}(T) + \mathsf{ker}(T^{\widetilde{\boldsymbol{a}}_e(T)}) \text{ is closed,} \\ &\Longleftrightarrow \quad \mathsf{Im}(T^m) + \mathsf{ker}(T^{m\,\widetilde{\boldsymbol{a}}_e(T^m)}) \text{ is closed,} \\ &\Longleftrightarrow \quad \mathsf{Im}(T^m) + \mathsf{ker}(T^{m\,\widetilde{\boldsymbol{a}}(T^m)}) \text{ is closed.} \end{aligned}$$

Consequently,

$$0 \in \varrho_{asc}^{g}(T) \Longleftrightarrow 0 \in \varrho_{asc}^{g}(T^m),$$

which completes the proof. ∎

Corollary 4.11. *Let $T \in \Upsilon(\mathsf{H})$ be a closed linear relation and let $P = (\lambda_1 - X)^{m_1}(\lambda_2 - X)^{m_2}\cdots(\lambda_n - X)^{m_n}$ be a complex polynomial such that $m_i \neq 0$ for every $i = 1, 2, \cdots, n$. Then*

$$0 \in \varrho_{asc}^{e,\,g}(P(T)) \Longleftrightarrow \lambda_i \in \varrho_{asc}^{e,\,g}(T), \quad \forall\, 1 \leq i \leq n$$

and

$$0 \in \varrho_{asc}^{g}(P(T)) \Longleftrightarrow \lambda_i \in \varrho_{asc}^{g}(T), \quad \forall\, 1 \leq i \leq n.$$

Proof. First, since $T \in \Upsilon(\mathsf{H})$ is closed, $P(T)$ is closed (see Remark 4.6). Now, from Theorem 4.9 and Theorem 4.10, we deduce

$$\begin{aligned} 0 \in \varrho_{asc}^{e,\,g}(P(T)) \quad &\Longleftrightarrow \quad 0 \in \bigcap_{1\leq i\leq n} \varrho_{asc}^{e,\,g}[(\lambda_i I - T)^{m_i}], \\ &\Longleftrightarrow \quad 0 \in \bigcap_{1\leq i\leq n} \varrho_{asc}^{e,\,g}(\lambda_i I - T), \\ &\Longleftrightarrow \quad \lambda_i \in \varrho_{asc}^{e,\,g}(T), \quad \forall\, 1 \leq i \leq n. \end{aligned}$$

In the same way, we prove that

$$0 \in \varrho^{g}_{asc}(P(T)) \Longleftrightarrow \lambda_i \in \varrho^{g}_{asc}(T), \quad \forall\, 1 \leq i \leq n,$$

and the proof is therefore complete. ∎

The following extends [3, Theorem 4.7], we do not require that the relation T be everywhere defined and $\varrho(T) \neq \emptyset$.

Theorem 4.12. *Let $T \in \Upsilon(\mathsf{H})$ be a closed linear relation and $P = (\lambda_1 - X)^{m_1}(\lambda_2 - X)^{m_2} \cdots (\lambda_n - X)^{m_n}$ be a complex polynomial such that $m_i \neq 0$ for all $i = 1, 2, \cdots, n$. Then*

$$P(\sigma^{e,g}_{asc}(T)) = \sigma^{e,g}_{asc}(P(T)) \quad \textit{and} \quad P(\sigma^{g}_{asc}(T)) = \sigma^{g}_{asc}(P(T)).$$

Proof. First note, by Remark 4.6, $P(T)$ is a closed linear relation. Now, from Corollary 4.11, it follows that

$$\begin{aligned} \lambda \in P(\sigma^{e,g}_{asc}(T)) &\Longleftrightarrow \lambda = P(\mu), \text{ where } \mu \in \sigma^{e,g}_{asc}(T), \\ &\Longleftrightarrow \lambda - P(Z) = (\mu - Z)^k Q(Z), \text{ where } Q(\mu) \neq 0, \\ &\Longleftrightarrow \lambda \in \sigma^{e,g}_{asc}(P(T)). \end{aligned}$$

The second equality, can be proved in the same way as the first one. This finishes the proof of the theorem. ∎

Example 4.13.

(*i*) Let T as in (*i*) of Example 4.5, we have $T \in \Upsilon(\mathcal{H})$ is a closed linear relation and $\mathcal{D}(T) \subsetneq \mathcal{H}$. Hence, if P is a non-constant complex polynomial, by [3, Theorem 4.7], it is not possible to deduce that $P(\sigma^{g}_{asc}(T)) = \sigma^{g}_{asc}(P(T))$ or $P(\sigma^{e,g}_{asc}(T)) = \sigma^{e,g}_{asc}(P(T))$. However, from Theorem 4.12, we can do this.

(*ii*) Let $B = (e_1, e_2, e_3)$ be a basis of $\mathbb{C}^3$. Consider the linear relation :

$$T\Big(\sum_{i=1}^{3} \alpha_i\, e_i\Big) = \mathsf{Vect}(e_1,\ e_2) + \alpha_3\, e_3, \ \ \forall\, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}.$$

Clearly, $T \in \Upsilon(\mathcal{H})$, because T is a closed linear relation everywhere defined and $T^n(0)$ is closed for all $n \in \mathbb{N}$. For all $\lambda \in \mathbb{C}$, we note that $0 \in (\lambda I - T)(e_1) = \mathsf{Vect}(e_1,\ e_2)$, this implies that $\varrho(T) = \emptyset$. Hence, if P is a non-constant complex polynomial, by [3, Theorem 4.7], it is not possible to deduce that $P(\sigma^{g}_{asc}(T)) = \sigma^{g}_{asc}(P(T))$ or $P(\sigma^{e,g}_{asc}(T)) = \sigma^{e,g}_{asc}(P(T))$. However, from Theorem 4.12, we can do this.

Remark 4.14. We note that Theorem 4.12 is false in general without the assumption that P is a non-constant polynomial, even if T is a bounded linear operator. For example, if H is a separable Hilbert space and $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of H, we define the following bounded operator on H,

$$T\Big(\sum_{n=0}^{+\infty} x_n e_n\Big) = \sum_{n=0}^{+\infty} \frac{x_n}{n+1} e_n.$$

Since $\mathsf{ker}(T) = \{0\}$ (which gives $\boldsymbol{a_e}(T) = 0$) and $\mathsf{Im}(T)$ is not closed in H, it follows that $0 \in \sigma^e_{asc}(T) \subseteq \sigma_{asc}(T)$. Put $P = c \in \mathbb{C}$, since $\sigma^e_{asc}(T)$ and $\sigma_{asc}(T)$ are non-empty sets, it follows that

$$P(\sigma^e_{asc}(T)) = P(\sigma_{asc}(T)) = \{c\}.$$

However, $\varrho_{asc}(P(T)) = \varrho_{asc}(cI) = \mathbb{C}$. Indeed, $\mathbb{C}\backslash\{c\} = \varrho(cI) \subseteq \varrho_{asc}(cI)$, and $cI - cI$ is the zero operator on H with ascent is equal to 1 and kernel is equal to H. Hence

$$\sigma^e_{asc}(P(T)) = \sigma_{asc}(P(T)) = \emptyset.$$

5. A spectral mapping theorems for essential g-descent and g-descent spectrums

We start this section with the following definitions. The descent and the essential descent resolvent sets of $T \in \mathcal{L_R}(\mathsf{H})$ are respectively defined by

$$\varrho_{des}(T) = \{\lambda \in \mathbb{C} : \boldsymbol{d}(T_\lambda) < +\infty\} \text{ and } \varrho^e_{des}(T) = \{\lambda \in \mathbb{C} : \ \boldsymbol{d_e}(T_\lambda) < +\infty\}.$$

The descent and the essential descent spectrums of T are respectively $\sigma_{des}(T) := \mathbb{C}\backslash\varrho_{des}(T)$ and $\sigma^e_{des}(T) := \mathbb{C}\backslash\varrho^e_{des}(T)$.

The g-descent resolvent set and the essential g-descent resolvent set of $T \in \mathcal{L_R}(\mathsf{H})$ are respectively defined by

$$\varrho^g_{des}(T) = \{\lambda \in \mathbb{C} : \widetilde{\boldsymbol{d}}(T_\lambda) < +\infty\} \text{ and } \varrho^{e,g}_{des}(T) = \{\lambda \in \mathbb{C} : \widetilde{\boldsymbol{d}}_{\boldsymbol{e}}(T_\lambda) < +\infty\}.$$

The g-descent and the essential g-descent spectrums of T are respectively $\sigma^g_{des}(T) := \mathbb{C}\backslash\varrho^g_{des}(T)$ and $\sigma^{e,g}_{des}(T) := \mathbb{C}\backslash\varrho^{e,g}_{des}(T)$. Evidently

$$\varrho(T) \subseteq \varrho^g_{des}(T) \subseteq \varrho_{des}(T) \subseteq \varrho^e_{des}(T), \ \ \varrho(T) \subseteq \varrho^g_{des}(T) \subseteq \varrho^{e,g}_{des}(T) \subseteq \varrho^e_{des}(T).$$

This section will be devoted to study the spectral mapping theorems of the g-descent and the essential g-descent spectrums of linear relations. The following lemmas will be used to prove the main result of this section.

Lemma 5.1. *Let $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{H})$, P and Q are relatively prime complex polynomials. Let $n \in \mathbb{N}\backslash\{0\}$, $A = P(T)$ and $B = Q(T)$. Then*

(*i*) $\widetilde{\boldsymbol{d}}_e(AB) = \max\{\widetilde{\boldsymbol{d}}_e(A),\, \widetilde{\boldsymbol{d}}_e(B)\}$ *and* $\widetilde{\boldsymbol{d}}(AB) = \max\{\widetilde{\boldsymbol{d}}(A),\, \widetilde{\boldsymbol{d}}(B)\}$,

(*ii*) *T possess a finite g-descent (resp. essential g-descent) if and only if the same holds for T^n.*

Proof. (*i*) Let $n \in \mathbb{N}$, from Lemma 4.8, we have

$$\begin{aligned}\mathsf{ker}(A^nB^n) + \mathsf{Im}(AB) &= \mathsf{ker}(A^n) + \mathsf{ker}(B^n) + \mathsf{Im}(A) \cap \mathsf{Im}(B),\\ &= \big(\mathsf{ker}(A^n) + \mathsf{Im}(A)\big) \cap \mathsf{Im}(B) + \mathsf{ker}(B^n),\\ &= \big(\mathsf{ker}(A^n) + \mathsf{Im}(A)\big) \cap \big(\mathsf{Im}(B) + \mathsf{ker}(B^n)\big).\end{aligned}$$

Therefore $\mathsf{ker}(A^nB^n) + \mathsf{Im}(AB) \subseteq \mathsf{ker}(A^n) + \mathsf{Im}(A) \subseteq \mathsf{H}$, and consequently,

$$\widetilde{\beta}_n(AB) = \widetilde{\beta}_n(A) + \dim\big(\mathsf{ker}(A^n) + \mathsf{Im}(A)\big)/\big([\mathsf{ker}(A^n) + \mathsf{Im}(A)] \cap [\mathsf{Im}(B) + \mathsf{ker}(B^n)]\big),$$

$$\max\{\widetilde{\beta}_n(A),\, \widetilde{\beta}_n(B)\} \leq \widetilde{\beta}_n(AB) \leq \widetilde{\beta}_n(A) + \widetilde{\beta}_n(B).$$

(*ii*) Let $m \in \mathbb{N}$ and S be the relation induced by T on $\mathsf{H}/\mathsf{ker}(T^{nm})$. Thus from [9, Lemma 5.1], we obtain

$$\widetilde{\beta}_{nm}(T) = \beta(S) \leq \beta(S^n) = \widetilde{\beta}_m(T^n) \leq n\,\beta(S) = n\,\widetilde{\beta}_{nm}(T).$$

This completes the proof. ∎

Lemma 5.2. *Let $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{H})$ and $m \in \mathbb{N}\backslash\{0\}$. Assume that A and B are defined as in* Lemma 5.1. Then

(*i*) $0 \in \varrho^{e,g}_{des}(T)$ if and only if $0 \in \varrho^{e,g}_{des}(T^m)$,

(*ii*) $0 \in \varrho^{g}_{des}(T)$ if and only if $0 \in \varrho^{g}_{des}(T^m)$,

(*iii*) $0 \in \varrho^{e,g}_{des}(AB)$ if and only if $0 \in \varrho^{e,g}_{des}(A) \cap \varrho^{e,g}_{des}(B)$,

(*iv*) $0 \in \varrho^{g}_{des}(AB)$ if and only if $0 \in \varrho^{g}_{des}(A) \cap \varrho^{g}_{des}(B)$.

Proof. It is an obvious consequence of Lemma 5.1. ∎

COROLLARY 5.3. *Let $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{H})$ and let $P = (\lambda_1 - X)^{m_1}(\lambda_2 - X)^{m_2} \cdots (\lambda_n - X)^{m_n}$ be a complex polynomial such that $m_i \neq 0$ for every $i = 1, 2, \cdots, n$. Then*

$$0 \in \varrho_{des}^{e,g}(P(T)) \Longleftrightarrow \lambda_i \in \varrho_{des}^{e,g}(T), \quad \forall\, 1 \leq i \leq n$$

and

$$0 \in \varrho_{des}^{g}(P(T)) \Longleftrightarrow \lambda_i \in \varrho_{des}^{g}(T), \quad \forall\, 1 \leq i \leq n.$$

Proof. From Lemma 5.2, it follows that

$$\begin{aligned} 0 \in \varrho_{des}^{e,g}(P(T)) &\Longleftrightarrow 0 \in \bigcap_{1 \leq i \leq n} \varrho_{des}^{e,g}[(\lambda_i I - T)^{m_i}], \\ &\Longleftrightarrow 0 \in \bigcap_{1 \leq i \leq n} \varrho_{des}^{e,g}(\lambda_i I - T), \\ &\Longleftrightarrow \lambda_i \in \varrho_{des}^{e,g}(T), \quad \forall\, 1 \leq i \leq n. \end{aligned}$$

In the same way, we obtain

$$0 \in \varrho_{des}^{g}(P(T)) \Longleftrightarrow \lambda_i \in \varrho_{des}^{g}(T), \quad \forall\, 1 \leq i \leq n.$$

The proof is complete. ∎

First note, that the results of this section are true for Banach spaces. Hence, the following extends [3, Theorem 3.4], we do not require that the relation T be everywhere defined and $\dim T(0) < +\infty$.

THEOREM 5.4. *Let $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{H})$ and $P = (\lambda_1 - X)^{\alpha_1} \cdots (\lambda_m - X)^{\alpha_m}$ be a complex polynomial such that $\alpha_i \neq 0$ for all $i = 1, 2, \cdots, m$. Then*

$$P(\sigma_{des}^{e,g}(T)) = \sigma_{des}^{e,g}(P(T)) \quad \text{and} \quad P(\sigma_{des}^{g}(T)) = \sigma_{des}^{g}(P(T)).$$

Proof. From Corollary 5.3, it follows that

$$\begin{aligned} \lambda \in P(\sigma_{des}^{e,g}(T)) &\Longleftrightarrow \lambda = P(\mu), \text{ where } \mu \in \sigma_{des}^{e,g}(T), \\ &\Longleftrightarrow \lambda - P(Z) = (\mu - Z)^k Q(Z), \text{ where } Q(\mu) \neq 0, \\ &\Longleftrightarrow \lambda \in \sigma_{des}^{e,g}(P(T)). \end{aligned}$$

The second equality, can be proved in the same way as the first one. This finishes the proof of the theorem. ∎

Example 5.5. Let T and K as in (i) of Example 4.5 (resp. suppose that $\dim K(0) = +\infty$ and we replace the condition $\mathcal{D}(K) \subsetneq \mathsf{H}$ by $\mathcal{D}(K) \subseteq \mathsf{H}$), we have $T \in \Upsilon(\mathcal{H})$ is a closed linear relation and $\mathcal{D}(T) \subsetneq \mathcal{H}$ (resp. $\dim T(0) = +\infty$). Hence, if P is a non-constant complex polynomial, by [3, Theorem 3.4], it is not possible to deduce that $P(\sigma^{g}_{des}(T)) = \sigma^{g}_{des}(P(T))$ (resp. $P(\sigma^{e,g}_{des}(T)) = \sigma^{e,g}_{des}(P(T))$). However, from Theorem 5.4, we can do this.

Remark 5.6. We note that Theorem 5.4 is false in general without the assumption that P is a non-constant polynomial, even if T is a bounded linear operator. For example, let T be defined as in Remark 4.14. Since $\mathsf{ker}(T) = \{0\}$ and $\mathsf{Im}(T)$ is not closed, then $\widetilde{\beta}_n(T) = \beta(T) = +\infty$, for all $n \in \mathbb{N}$, and thus $0 \in \sigma^{e,g}_{des}(T) \subseteq \sigma^{g}_{des}(T)$. Put $P = c \in \mathbb{C}$. Since $\sigma^{e,g}_{des}(T)$ and $\sigma^{g}_{des}(T)$ are non-empty sets, it follows that

$$P(\sigma^{e,g}_{des}(T)) = P(\sigma^{g}_{des}(T)) = \{c\}.$$

However, $\varrho^{g}_{des}(P(T)) = \varrho^{g}_{des}(cI) = \mathbb{C}$. Indeed, $\mathbb{C}\backslash\{c\} = \varrho(cI) \subseteq \varrho^{g}_{des}(cI)$, and $cI - cI$ is the zero operator with g-descent is equal to 1. Therefore

$$\sigma^{g}_{des}(P(T)) = \sigma^{e,g}_{des}(P(T)) = \emptyset.$$

6. Decomposition theorems

First observe that if $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{H})$ is a range space relation such that $\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T) < +\infty$ and $\mathsf{Im}(T) + \mathsf{ker}(T^{\widetilde{a}_e(T)})$ is closed in H, then T is a quasi-Fredholm relation (see, [7, Definition 5.1]). In the following, we prove a decomposition theorem of linear relations with finite essential g-ascent such that $\mathsf{Im}(T) + \mathsf{ker}(T^{\widetilde{a}_e(T)})$ is closed in H.

Theorem 6.1. *Let $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$. Then there exists $n \in \mathbb{N}$ such that $\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T) \leq n$ and $\mathsf{Im}(T) + \mathsf{ker}(T^n)$ is closed in H if and only if there exist $d \in \mathbb{N}$ and two closed subspaces M and N such that :*

(i) $\mathsf{H} = \mathsf{M} \dotplus \mathsf{N}$;

(ii) $\mathsf{Im}(T^d) \subseteq \mathsf{M}$, $T(\mathsf{M}) \subseteq \mathsf{M}$, $\mathsf{N} \subseteq \mathsf{ker}(T^d)$ *and, if* $d > 0$, $\mathsf{N} \not\subset \mathsf{ker}(T^{d-1})$;

(iii) $\mathsf{G}(T) = [\mathsf{G}(T) \cap (\mathsf{M} \times \mathsf{M})] \dotplus [\mathsf{G}(T) \cap (\mathsf{N} \times \mathsf{N})]$;

(iv) *the restriction of T to M is both upper semi-Fredholm and regular relation;*

(v) $A \in \mathcal{L}_{\mathcal{R}}(\mathsf{N})$ *such that its graph is the subspace* $\mathsf{G}(T) \cap (\mathsf{N} \times \mathsf{N})$, *then* A *is a bounded operator everywhere defined and* $\mathsf{G}(A^d) = \mathsf{N} \times \{0\}$.

Proof. " $\Longrightarrow$ " First, the assertions (i)-(iii) and (v) follow from [7, Theorem 5.2], and by the same theorem we know that $S = T_{|\mathsf{M}}$ the restriction of T to M is regular. Let $m = \max\{d,\ n\}$, since S is regular and $\mathsf{Im}(T^m) \subseteq \mathsf{M}$, it follows that

$$\mathsf{ker}(T) \cap \mathsf{Im}(T^m) = \mathsf{ker}(S) \cap \mathsf{Im}(S^m) = \mathsf{ker}(S),$$

and hence $\alpha(S) = \widetilde{\alpha}_m(T) < +\infty$. Therefore $S \in \Phi_+(\mathsf{M})$.

" $\Longleftarrow$ " Let $S = T_{|\mathsf{M}}$ be the restriction of T to M and $A \in \mathcal{L}_{\mathcal{R}}(\mathsf{N})$ such that $\mathsf{G}(A) = \mathsf{G}(T) \cap (\mathsf{N} \times \mathsf{N})$, so

$$\mathsf{ker}(T) \cap \mathsf{Im}(T^d) = \mathsf{ker}(S) \cap \mathsf{Im}(S^d) = \mathsf{ker}(S) \ \text{ and } \ \widetilde{\alpha}_d(T) < +\infty.$$

This implies that $\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T) \leq d$. By [7, Theorem 6.4], $\mathsf{Im}(T) + \mathsf{ker}(T^d)$ is closed and from [6, Lemma 2.9], it follows that $\mathsf{Im}(T) + \mathsf{ker}(T^n)$ is closed for all $n \geq \widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T)$, which completes the proof. ∎

Now from [6, Lemma 2.9], we know that if $\widetilde{\boldsymbol{a}}(T) = n < +\infty$ and $\mathsf{Im}(T) + \mathsf{ker}(T^n)$ is closed, then $\widetilde{\boldsymbol{a}}_{\boldsymbol{e}}(T) \leq n$ and $\mathsf{Im}(T) + \mathsf{ker}(T^n)$ is closed. So, as a consequence of Theorem 6.1, we obtain the following theorem.

THEOREM 6.2. *Let* $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$. *Then there exists* $n \in \mathbb{N}$ *such that* $\widetilde{\boldsymbol{a}}(T) = n < +\infty$ *and* $\mathsf{Im}(T) + \mathsf{ker}(T^n)$ *is closed if and only if there exist* $d \in \mathbb{N}$ *and two closed subspaces* M *and* N *such that :*

(i) $\mathsf{H} = \mathsf{M} \dotplus \mathsf{N}$,

(ii) $T(\mathsf{M} \cap \mathcal{D}(T)) \subseteq \mathsf{M}$, $\mathsf{Im}(T^d) \subseteq \mathsf{M}$, $\mathsf{N} \subseteq \mathsf{ker}(T^d)$ *and, if* $d > 0$, $\mathsf{N} \not\subset \mathsf{ker}(T^{d-1})$,

(iii) $\mathsf{G}(T) = [\mathsf{G}(T) \cap (\mathsf{M} \times \mathsf{M})] \dotplus [\mathsf{G}(T) \cap (\mathsf{N} \times \mathsf{N})]$,

(iv) *the restriction of* T *to* M *is injective with closed range,*

(v) *if* $A \in \mathcal{L}_{\mathcal{R}}(\mathsf{N})$ *such that its graph is the subspace* $\mathsf{G}(T) \cap (\mathsf{N} \times \mathsf{N})$, *then* A *is a bounded operator everywhere defined and* $\mathsf{G}(A^d) = \mathsf{N} \times \{0\}$.

The following lemma will be needed in the proof of Theorem 6.4.

Lemma 6.3. *Let $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$ such that $\boldsymbol{d_e}(T) < +\infty$. The following statements are equivalent :*

(*i*) $\mathsf{Im}(T^n) \cap \mathsf{ker}(T)$ *is closed for some* $n \geq \boldsymbol{d_e}(T)$,

(*ii*) $\mathsf{Im}(T^n) \cap \mathsf{ker}(T)$ *is closed for all* $n \geq \boldsymbol{d_e}(T)$.

Proof. It is clear that only the implication "$(i) \Longrightarrow (ii)$" needs to prove. Let $n_0 \geq \boldsymbol{d_e}(T)$ such that $\mathsf{Im}(T^{n_0}) \cap \mathsf{ker}(T)$ is closed. First, we prove that $\mathsf{Im}(T^{n_0+1}) \cap \mathsf{ker}(T)$ is closed. By the equality (1) in the proof of Lemma 2.4, we get

$$\beta_{n_0}(T) \geq S_{n_0}(T) = \dim \big(\mathsf{Im}(T^{n_0}) \cap \mathsf{ker}(T)\big)/\big(\mathsf{Im}(T^{n_0+1}) \cap \mathsf{ker}(T)\big),$$

and hence from [7, Propositions 3.9, 3.10 and 4.8, Lemma 4.2], $\mathsf{Im}(T^{n_0+1}) \cap \mathsf{ker}(T)$ is closed. Now if $n_0 > \boldsymbol{d_e}(T)$, then $n_0 - 1 \geq \boldsymbol{d_e}(T)$ and so $\dim \big(\mathsf{Im}(T^{n_0-1}) \cap \mathsf{ker}(T)\big)/\big(\mathsf{Im}(T^{n_0}) \cap \mathsf{ker}(T)\big) < +\infty$. Therefore $\mathsf{Im}(T^{n_0-1}) \cap \mathsf{ker}(T)$ is also closed. This completes the proof. ∎

In the following result, we prove a decomposition theorem for $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$, with $n = \widetilde{\boldsymbol{d}}_{\boldsymbol{e}}(T) < +\infty$ and $\mathsf{Im}(T^n) \cap \mathsf{ker}(T)$ is closed in H.

Theorem 6.4. *Let $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$. Then $\widetilde{\boldsymbol{d}}_{\boldsymbol{e}}(T) \leq n$ and $\mathsf{Im}(T^n) \cap \mathsf{ker}(T)$ is closed for some $n \in \mathbb{N}$ if and only if there exist $d \in \mathbb{N}$ and two closed subspaces M and N such that :*

(*i*) $\mathsf{H} = \mathsf{M} \dotplus \mathsf{N}$,

(*ii*) $T(\mathsf{M} \cap \mathcal{D}(T)) \subseteq \mathsf{M}$, $\mathsf{Im}(T^d) \subseteq \mathsf{M}$, $\mathsf{N} \subseteq \mathsf{ker}(T^d)$ *and, if* $d > 0$, $\mathsf{N} \not\subseteq \mathsf{ker}(T^{d-1})$,

(*iii*) $\mathsf{G}(T) = [\mathsf{G}(T) \cap (\mathsf{M} \times \mathsf{M})] \dotplus [\mathsf{G}(T) \cap (\mathsf{N} \times \mathsf{N})]$,

(*iv*) *the restriction of T to M is both lower semi-Fredholm and regular relation,*

(*v*) *if $A \in \mathcal{L}_{\mathcal{R}}(\mathsf{N})$ such that its graph is the subspace $\mathsf{G}(T) \cap (\mathsf{N} \times \mathsf{N})$, then A is a bounded operator everywhere defined and $\mathsf{G}(A^d) = \mathsf{N} \times \{0\}$.*

Proof. "$\Longrightarrow$" First, from Lemma 2.4, we have $\widetilde{\boldsymbol{d}}_{\boldsymbol{e}}(T) \leq p(T) < +\infty$ and $\widetilde{\beta}_{p(T)}(T)$ is finite, which implies that $\mathsf{Im}(T) + \mathsf{ker}(T^{p(T)})$ is closed in H. Moreover, by Lemma 6.3, T is a quasi-Fredholm relation and so from [7, Theorem 5.2], it follows that there exist $d \in \mathbb{N}$ and two closed subspaces M and N such that $\mathsf{H} = \mathsf{M} \dotplus \mathsf{N}$, $T(\mathsf{M} \cap \mathcal{D}(T)) \subseteq \mathsf{M}$, $\mathsf{N} \subseteq \mathsf{ker}(T^d) \subseteq \mathcal{D}(T)$, if $d > 0$, $\mathsf{N} \not\subset \mathsf{ker}(T^{d-1})$ and the restriction of T to M, $S = T_{|\mathsf{M}}$, is regular. Now, let $m = \max\{d,\ n\}$, then $\mathsf{Im}(T) + \mathsf{ker}(T^m) = \mathsf{Im}(S) \dotplus \mathsf{N}$ (see the equality (6.7) in the proof of [7, Theorem 6.4]). This implies that

$$\dim \mathsf{M}/\mathsf{Im}(S) = \dim[\mathsf{M} \dotplus \mathsf{N}]/[\mathsf{Im}(S) \dotplus \mathsf{N}] = \widetilde{\beta}_m(T) < +\infty$$

and consequently $S \in \Phi_-(\mathsf{M})$.

" $\Longleftarrow$ " Let $S = T_{|\mathsf{M}}$ be the restriction of T to M and $A \in \mathcal{L}_{\mathcal{R}}(\mathsf{N})$ such that $\mathsf{G}(A) = \mathsf{G}(T) \cap (\mathsf{N} \times \mathsf{N})$, so that

$$\mathsf{Im}(T) + \mathsf{ker}(T^d) = \mathsf{Im}(S) \dotplus \mathsf{N}, \quad \mathsf{ker}(T) \cap \mathsf{Im}(T^d) = \mathsf{ker}(S) \cap \mathsf{Im}(S^d) = \mathsf{ker}(S).$$

This implies that $\widetilde{\beta}_d(T) \leq \beta(S) < +\infty$ and $\mathsf{ker}(T) \cap \mathsf{Im}(T^d)$ is closed. This completes the proof of the theorem. ∎

Now from Lemma 6.3, we know that if $\widetilde{\boldsymbol{d}}(T) = n < +\infty$ and $\mathsf{Im}(T^n) \cap \mathsf{ker}(T)$ is closed, then $\widetilde{\boldsymbol{d}}_{\boldsymbol{e}}(T) \leq n$ and $\mathsf{Im}(T^n) \cap \mathsf{ker}(T)$ is closed. Therefore, we can prove the following theorem similarly as Theorem 6.4.

Theorem 6.5. *Let $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$. Then $\widetilde{\boldsymbol{d}}(T) \leq n$ and $\mathsf{Im}(T^n) \cap \mathsf{ker}(T)$ is closed for some $n \in \mathbb{N}$ if and only if there exist $d \in \mathbb{N}$ and two closed subspaces M and N such that :*

- *(i)* $\mathsf{H} = \mathsf{M} \dotplus \mathsf{N}$,
- *(ii)* $T(\mathsf{M} \cap \mathcal{D}(T)) \subseteq \mathsf{M}$, $\mathsf{Im}(T^d) \subseteq \mathsf{M}$, $\mathsf{N} \subseteq \mathsf{ker}(T^d)$ *and, if* $d > 0$, $\mathsf{N} \not\subset \mathsf{ker}(T^{d-1})$,
- *(iii)* $\mathsf{G}(T) = [\mathsf{G}(T) \cap (\mathsf{M} \times \mathsf{M})] \dotplus [\mathsf{G}(T) \cap (\mathsf{N} \times \mathsf{N})]$,
- *(iv) the restriction of T to M is surjective,*
- *(v) if $A \in \mathcal{L}_{\mathcal{R}}(\mathsf{N})$ such that its graph is the subspace $\mathsf{G}(T) \cap (\mathsf{N} \times \mathsf{N})$, then A is a bounded operator everywhere defined and $\mathsf{G}(A^d) = \mathsf{N} \times \{0\}$.*

Remark 6.6. Let $T \in \mathcal{C}_{\mathcal{R}}(\mathsf{H})$ such that $\max\{\widetilde{\boldsymbol{a}}(T),\ \widetilde{\boldsymbol{d}}(T)\} \leq m$, for some $m \in \mathbb{N}$. It is clear that $S = T_{|\mathsf{Im}(T^m)}$, the restriction of T to $\mathsf{Im}(T^m)$ is bijective, $\mathsf{H} = \mathsf{Im}(T^m) + \mathsf{ker}(T^m)$ and $\mathsf{Im}(T^m) \cap \mathsf{ker}(T^m) = \{0\}$ (see the equality (1) in the proof of Lemma 3.7), so from [7, Propositions 3.10, 4.8, Lemma 4.2], it follows that $\mathsf{Im}(T^m)$ and $\ker(T^m)$ are both closed. Let $A \in \mathcal{L}_{\mathcal{R}}(\ker(T^m))$ such that $\mathsf{G}(A) = \mathsf{G}(T) \cap (\ker(T^m) \times \ker(T^m))$. First note that $A(0) \subseteq \ker(T^m) \cap \mathsf{Im}(T^m) = \{0\}$ and $\mathsf{G}(A)$ is closed, which implies that A is a closed operator. Since

$$\begin{aligned}\mathcal{D}(A) &= \{x \in \ker(T^m) : Ax \neq \emptyset\} \\ &= \{x \in \ker(T^m) : Tx \neq \emptyset \text{ in } \ker(T^m)\} \\ &= \{x \in \ker(T^m) : \exists\, y \in Tx \text{ and } y \in \ker(T^m)\} \\ &= \{x \in \ker(T^m) : \exists\, y \in Tx \text{ and } 0 \in T^m(y)\} \\ &= \{x \in \ker(T^m) : 0 \in T^{m+1}(x)\} \\ &= \{x \in \ker(T^m) : x \in \ker(T^{m+1})\} \\ &= \ker(T^m),\end{aligned}$$

then A is a bounded operator everywhere defined. But $A^m(\ker(T^m)) \subseteq \ker(T^m) \cap \mathsf{Im}(T^m) = \{0\}$, so $\mathsf{G}(A^m) = \mathsf{G}(T^m) \cap (\ker(T^m) \times \ker(T^m)) = \ker(T^m) \times \{0\}$. Now, we will show that $\mathsf{G}(T) = [\mathsf{G}(T) \cap (\mathsf{Im}(T^m) \times \mathsf{Im}(T^m))] \dotplus [\mathsf{G}(T) \cap (\ker(T^m) \times \ker(T^m))]$. Let $(x, y) \in \mathsf{G}(T)$, then $x = x_1 + x_2$ with $x_1 \in \mathsf{Im}(T^m)$ and $x_2 \in \ker(T^m)$. Therefore, there exist $y_1 \in T(x_1) \subseteq \mathsf{Im}(T^m)$ and $y_2 \in T(x_2)$ such that $y = y_1 + y_2$. Clearly,

$$\begin{aligned}y_2 \in T(x_2) \subseteq T\big(\ker(T^m)\big) &= T\big(\ker(T^{m+1})\big) = TT^{-1}\big(\ker(T^m)\big) \\ &= \ker(T^m) \cap \mathsf{Im}(T) + T(0) \subseteq \ker(T^m) + T(0)\end{aligned}$$

and hence $y_2 = y_2' + y_2''$, for some $y_2' \in \ker(T^m)$ and $y_2'' \in T(0)$. We have,

$$y_1 + y_2'' \in T(x_1) + T(0) = T(x_1) \subseteq \mathsf{Im}(T^m)$$

and $y_2' = y_2 - y_2'' \in T(x_2) + T(0) = T(x_2)$, so

$$\begin{aligned}(x, y) = (x_1, y_1 + y_2'') + (x_2, y_2') \in{}& [\mathsf{G}(T) \cap (\mathsf{Im}(T^m) \times \mathsf{Im}(T^m))] \\ &\dotplus [\mathsf{G}(T) \cap (\ker(T^m) \times \ker(T^m))].\end{aligned}$$

This implies that

$$\mathsf{G}(T) = [\mathsf{G}(T) \cap (\mathsf{Im}(T^m) \times \mathsf{Im}(T^m))] \dotplus [\mathsf{G}(T) \cap (\ker(T^m) \times \ker(T^m))].$$

Finally, if we put $\mathsf{M} = \mathsf{Im}(T^m)$ and $\mathsf{N} = \ker(T^m)$, then M and N satisfy the conditions (i)-(v) of Theorems 6.2 and 6.5.

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