

## On Additive Preservers of Certain Classes of Algebraic Operators

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*Abstract:* In this article we provide a complete description of all additive surjective unital maps in the algebra of all bounded linear operators acting on an infinite-dimensional Hilbert space, preserving in both directions the set of non-invertible algebraic operators or the set of invertible algebraic operators.

*Key words:* Algebraic operators, Linear preserver problems.

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### 1. INTRODUCTION

Throughout this paper,  $X$  denotes an infinite-dimensional Banach space over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , and  $\mathcal{B}(X)$  the algebra of all bounded linear operators on  $X$ . Let  $\Lambda$  be a subset of  $\mathcal{B}(X)$ . We shall say that a map  $\Phi : \mathcal{B}(X) \rightarrow \mathcal{B}(X)$  preserves  $\Lambda$  in both directions if for every bounded operator  $T$  on  $X$ ,

$$T \in \Lambda \text{ if and only if } \Phi(T) \in \Lambda.$$

The problem of determining linear maps on  $\mathcal{B}(X)$  that leave invariant certain subsets has attracted the attention of many mathematicians, and numerous results revealing the algebraic structure of such maps are obtained in the last few decades. For excellent expositions on linear preserver problems, the reader is referred to [1, 2, 3, 7, 9, 11] and the references therein.

Let  $\mathcal{B}_o(X)$  denote the linear span of the set of all nilpotent operators in  $\mathcal{B}(X)$ . Šemrl showed in [16] that a surjective linear map  $\Phi : \mathcal{B}_o(X) \rightarrow \mathcal{B}_o(X)$  is nilpotent preserving in both directions if and only if it has one of the following two forms

$$T \mapsto cATA^{-1} \quad \text{or} \quad T \mapsto cAT'A^{-1}$$

where  $c$  is a non-zero complex number,  $A$  is a bounded bijective linear operator between suitable spaces and  $T'$  denotes the adjoint of  $T$  on the topological dual space  $X'$  of  $X$ . In the special case of Hilbert space  $H$ , this result is extended to linear maps  $\Phi$  defined on the whole space  $\mathcal{B}(H)$ .

In [2], the authors proved that if  $\Phi : \mathcal{B}(X) \rightarrow \mathcal{B}(X)$  is a surjective linear map which is continuous in the weak operator topology and preserves idempotent operators then it has one of the following two forms

$$T \mapsto ATA^{-1} \quad \text{or} \quad T \mapsto AT'A^{-1}$$

where  $A$  is a bounded bijective linear operator between suitable spaces. In the context of infinite-dimensional complex Hilbert space  $H$ , the same authors obtained in [3] a complete description of bijective linear maps on  $\mathcal{B}(H)$  preserving idempotent operators.

Recall that an operator  $T \in \mathcal{B}(X)$  is *algebraic* if there exists a non-zero complex polynomial  $P$  such that  $P(T) = 0$ . Since nilpotent operators and idempotent operators are algebraic, the question arises as to what can we say about surjective linear maps preserving algebraic operators in both directions. This question is still open. However, partial answers are given. More precisely, let  $P$  be a complex polynomial with  $\deg(P) \geq 2$ . It was shown by Šemrl [17] that if  $\Phi$  is a unital (that is,  $\Phi(I) = I$ ) surjective linear map preserving the set of all operators annihilated by  $P$  in both directions, then  $\Phi$  takes one of the following two forms

$$T \mapsto ATA^{-1} \quad \text{or} \quad T \mapsto AT^{\text{tr}}A^{-1}$$

where  $A \in \mathcal{B}(H)$  is bijective and  $T^{\text{tr}}$  denotes the transpose of  $T$  relative to a fixed but arbitrary orthonormal basis. Later, J. Hou and S. Hou determined in [8] under additional conditions the general form of linear maps  $\Phi$  preserving operators annihilated by  $P$  in only one direction, that is,  $P(\Phi(T)) = 0$  whenever  $P(T) = 0$ .

Let us introduce the following subsets:

- (1)  $\mathcal{A}_s(X)$  the set of non-invertible algebraic operators in  $\mathcal{B}(X)$ ;
- (2)  $\mathcal{A}_i(X)$  the set of invertible algebraic operators in  $\mathcal{B}(X)$ ;
- (3)  $\mathcal{A}_o(X)$  the linear span of algebraic operators in  $\mathcal{B}(X)$ .

It is worth mentioning that  $\mathcal{A}_s(X)$  is closely related to several subsets studied in the literature in connection with linear preserver problems:

- (1) the set of all rank one operators [13],
- (2) the set of all nilpotent operators [1, 16],
- (3) the set of all idempotent operators except  $I$  [2, 3],
- (4) the set of all potent operators except  $I$  [14].

The aim of this paper is to determine the structure of additive mappings  $\Phi : \mathcal{B}(X) \rightarrow \mathcal{B}(X)$  preserving  $\mathcal{A}_s(X)$  (resp.  $\mathcal{A}_i(X)$ ) in both directions. More precisely the main result can be summarized in the following theorem:

**THEOREM A.** Let  $\Phi : \mathcal{B}(X) \rightarrow \mathcal{B}(X)$  be an additive surjective unital map. Then the following assertions are equivalent:

- (i)  $\Phi$  preserves the set of non-invertible algebraic operators in both directions;
- (ii)  $\Phi$  preserves the set of invertible algebraic operators in both directions;
- (iii) there exists either a bounded invertible linear, or conjugate linear, operator  $A : X \rightarrow X$  such that

$$\Phi(T) = ATA^{-1} \quad \text{for all } T \in \mathcal{A}_o(X),$$

or, a bounded invertible linear, or conjugate linear, operator  $B : X' \rightarrow X$  such that

$$\Phi(T) = BT'B^{-1} \quad \text{for all } T \in \mathcal{A}_o(X).$$

It follows from Theorem A that if  $\Phi$  preserves non-invertible algebraic operators in both directions then it preserves also algebraic operators in both directions. The converse is generally false as can be shown by considering the map

$$\Phi(T) = T + \psi(T)I \quad \text{for all } T \in \mathcal{B}(X),$$

where  $\psi$  is a linear form on  $\mathcal{B}(X)$  such that  $\psi(I) = 0$  and  $\psi(N) \neq 0$  for some nilpotent operator  $N \in \mathcal{B}(X)$ .

No continuity of  $\Phi$  was assumed in the above theorem. Nevertheless, we get as a result that both  $A, A^{-1}, B$  and  $B^{-1}$  are bounded, and consequently,  $\Phi$  is continuous.

Let  $H$  be an infinite-dimensional Hilbert space. For  $T \in \mathcal{B}(H)$  let  $T^* : H \rightarrow H$  be the Hilbert space adjoint. Note that  $T^* = J_o^{-1}T'J_o$  where  $J_o : H \rightarrow H'$  is the natural conjugate linear mapping defined by

$$J_o(x)(u) = \langle u, x \rangle \quad \text{for all } x, u \in H.$$

By a result of Pearcy and Topping [15], every operator in  $\mathcal{B}(H)$  is a finite sum of square-zero operators, which implies that  $\mathcal{A}_o(H) = \mathcal{B}(H)$ . Hence, we obtain the following theorem as an immediate consequence of the previous one.

**THEOREM B.** Let  $H$  be an infinite-dimensional Hilbert space, and let  $\phi : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$  be an additive surjective unital map. The following assertions are equivalent:

- (i)  $\Phi$  preserves the set of non-invertible algebraic operators in both directions;
- (ii)  $\Phi$  preserves the set of invertible algebraic operators in both directions;
- (iii) there exists an invertible bounded linear, or conjugate linear, operator  $A$  on  $H$  such that either

$$\Phi(T) = ATA^{-1} \quad \text{for all } T \in \mathcal{B}(H),$$

or

$$\Phi(T) = AT^*A^{-1} \quad \text{for all } T \in \mathcal{B}(H).$$

Recall that  $X$  is a Banach space of infinite multiplicity if  $X$  is isomorphic to  $X \oplus X$  and to  $X \oplus X \oplus \cdots$  an infinite direct sum  $X^{(\infty)}$  of copies of  $X$  in such a way that arbitrary permutations of coordinates and transformations of the form  $A^{(\infty)}$ , with  $A \in \mathcal{B}(X)$ , are bounded operators, and such that, for  $A, B \in \mathcal{B}(X)$ ,  $A^{(\infty)} \oplus B^{(\infty)}$  acting on  $X^{(\infty)} \oplus X^{(\infty)}$  is similar to  $(A + B)^{(\infty)}$  acting on  $(X \oplus X)^{(\infty)}$ . Infinite-dimensional  $l_p$ -space,  $L_p[0, 1]$ ,  $1 \leq p \leq \infty$ , and infinite-dimensional Hilbert spaces are examples of Banach spaces of infinite multiplicity, see [5].

Now, if  $X$  is a Banach space with infinite multiplicity, then we can give a complete classification of all additive surjective unital maps on  $\mathcal{B}(X)$  preserving in both directions  $\mathcal{A}_s(X)$  or  $\mathcal{A}_i(X)$ . Indeed, it follows from [1, Lemma 2.9] that every operator in  $\mathcal{B}(X)$  can be written as a sum of at most 8 square zero operators, which implies that  $\mathcal{A}_o(X) = \mathcal{B}(X)$ . Hence, we obtain the following result as a consequence of Theorem A.

*Remark 1.1.* Let  $X$  be a Banach space with infinite multiplicity, and let  $\phi : \mathcal{B}(X) \rightarrow \mathcal{B}(X)$  be an additive surjective unital map. The following assertions are equivalent:

- (i)  $\Phi$  preserves the set of non-invertible algebraic operators in both directions;
- (ii)  $\Phi$  preserves the set of invertible algebraic operators in both directions;
- (iii) there exists either a bounded invertible linear, or conjugate linear, operator  $A : X \rightarrow X$  such that

$$\Phi(T) = ATA^{-1} \quad \text{for all } T \in \mathcal{B}(X),$$

or, a bounded invertible linear, or conjugate linear, operator  $B : X' \rightarrow X$  such that

$$\Phi(T) = BT'B^{-1} \quad \text{for all } T \in \mathcal{B}(X).$$

It should be noted that this result holds true also if we suppose that  $X$  is an arbitrary infinite-dimensional Banach space and  $\Phi$  is an additive surjective unital map on  $\mathcal{B}(X)$  which is continuous in the weak operator topology. Indeed, it is well-known that the sub-algebra  $\mathcal{F}(X)$  of all finite rank bounded operators on  $X$  is dense in the weak operator topology in  $\mathcal{B}(X)$ . In particular,  $\mathcal{A}_0(X)$  is dense in the weak operator topology in  $\mathcal{B}(X)$ .

The paper is organized as follows. In the second section, we present useful results on the perturbation of algebraic non-invertible operators. Especially, we provide a characterization of rank one operators via the algebraic invertible operators. The last section is devoted to the proof of the main theorem.

## 2. ALGEBRAIC NON-INVERTIBLE OPERATORS

For an operator  $T \in \mathcal{B}(X)$ , write  $\ker(T)$  for its kernel,  $\text{ran}(T)$  for its range and  $\sigma(T)$  for its spectrum.

Let  $z \in X$  and  $f \in X'$  be non-zero. The symbol  $z \otimes f$  stands for the rank one operator defined by  $(z \otimes f)(x) = f(x)z$  for all  $x \in X$ . Note that every rank one operator in  $\mathcal{B}(X)$  can be written in this form, and the spectrum of such operator is  $\sigma(z \otimes f) = \{0, f(z)\}$ .

*Remark 2.1.* Let  $T$  be a bounded operator on  $X$ . The following well-known properties will be often used in the sequel:

- (i) Let  $z \in X$  and  $f \in X'$ . If  $T$  is invertible then it follows easily from [9, Lemma 4] that

$$T + z \otimes f \text{ is non-invertible if and only if } f(T^{-1}z) = -1. \quad (2.1)$$

(ii) For every finite rank operator  $F$ , it follows from [4, Proposition 3.6] that

$T$  is algebraic if and only if  $T + F$  is algebraic.

(iii)  $T$  is algebraic if and only if  $T'$  is algebraic.

(iv) For every invertible bounded linear, or conjugate linear, operator  $A$  on  $X$ ,

$T$  is algebraic if and only if  $ATA^{-1}$  is algebraic.

(v) If  $T$  is algebraic then  $\sigma(T)$  is finite and formed by the poles of the resolvent of  $T$  of finite order, see [4], and in this case we have also  $\sigma(T) = \sigma_p(T)$  where  $\sigma_p(T)$  stands for the point spectrum of  $T$ , see [18, Theorem V.11.3].

**PROPOSITION 2.2.** *Let  $T \in \mathcal{A}_s(X)$ , and let  $F \in \mathcal{B}(X)$  be a rank one operator such that  $T + F \in \mathcal{A}_s(X)$ . Then  $T + 2F \in \mathcal{A}_s(X)$ .*

*Proof.* Put  $F = z \otimes f$  where  $z \in X$  and  $f \in X'$  are non-zero. Since  $T$  is algebraic then so is  $T + 2F$ , and hence it suffices to show that  $T + 2F$  is not invertible.

Clearly, if  $\ker(T) \cap \ker(f) \neq \{0\}$  then  $\ker(T + 2F)$  is non-trivial, and so  $T + 2F \notin \mathcal{A}_s(X)$ . Assume that  $\ker(T) \cap \ker(f) = \{0\}$ . Since  $T + F \in \mathcal{A}_s(X)$ , it follows by Remark 2.1 (v) that  $T + F$  is not injective. Let  $u \in \ker(T + F)$  be non-zero. Then  $Tu = -f(u)z$ , and hence  $f(u) \neq 0$ , since otherwise  $u \in \ker(T) \cap \ker(f)$ , a contradiction. Let  $v = -f(u)^{-1}u$ . Then

$$f(v) = -1, \quad z = Tv \quad \text{and} \quad I + 2v \otimes f \text{ is invertible.}$$

Consequently,  $T + 2F$  is not invertible because  $T + 2F = T(I + 2v \otimes f)$  and  $T$  is not invertible. This completes the proof. ■

The following example shows both that  $\mathcal{A}_s(X)$  is not stable under rank one perturbations, and that the assumption  $T + F \in \mathcal{A}_s(X)$  in Proposition 2.2 is necessary.

**EXAMPLE 2.3.** Let  $T \in \mathcal{B}(X)$  be any projection with  $\text{codim ran}(T) = 1$ . Then  $F = I - T$  is a rank one operator. Moreover, for every non-zero  $\alpha \in \mathbb{C}$ , it follows that  $T + \alpha F$  is algebraic, by Remark 2.1 (ii), and invertible with inverse  $T + \alpha^{-1}F$ . Consequently,  $T + F$  and  $T + 2F$  do not belong to  $\mathcal{A}_s(X)$ .

The following theorem, which is interesting in itself, will play a crucial role in proving the main result.

**THEOREM 2.4.** *Let  $F \in \mathcal{A}_s(X)$  be non-zero. Then the following assertions hold:*

- (i) *there exists  $T \in \mathcal{A}_s(X)$  such that  $T + 2F \notin \mathcal{A}_s(X)$ ;*
- (ii) *if  $\dim \text{ran}(F) \geq 2$  then there exists  $T \in \mathcal{A}_s(X)$  such that  $T + F \in \mathcal{A}_s(X)$  and  $T + 2F \notin \mathcal{A}_s(X)$ .*

Before proving this theorem, we need to establish the following lemma.

**LEMMA 2.5.** *Let  $Y$  and  $Z$  be two non-trivial closed subspaces such that  $X = Y \oplus Z$ , and let  $T \in \mathcal{B}(X)$  be the operator represented by the matrix*

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$

*Then  $T$  is algebraic if and only if  $A$  and  $C$  are algebraic. Furthermore, in this case, we have  $\sigma(T) = \sigma(A) \cup \sigma(C)$ .*

*Proof.* Note that for every complex polynomial  $q$ , one can easily verify that

$$q(T) = \begin{pmatrix} q(A) & B_q \\ 0 & q(C) \end{pmatrix}, \tag{2.2}$$

for some bounded operator  $B_q$ .

Suppose that  $T$  is algebraic and let  $q$  be a non-zero complex polynomial such that  $q(T) = 0$ . It follows from (2.2) that  $q(A) = q(C) = 0$ . Thus,  $A$  and  $C$  are algebraic.

Conversely, assume that  $A$  and  $C$  are algebraic. Let  $p$  be a non-zero complex polynomial such that  $p(A) = p(C) = 0$ . Then

$$p(T) = \begin{pmatrix} 0 & B_p \\ 0 & 0 \end{pmatrix},$$

and since  $p(T)^2 = 0$ , we get that  $T$  is algebraic.

Finally,  $\sigma(T) = \sigma(A) \cup \sigma(C)$  follows immediately from [6, Corollary 8], which completes the proof. ■

*Proof of Theorem 2.4.* Let us establish first the second assertion. Since  $\dim \text{ran}(F) \geq 2$ , there exist two vectors  $a, b \in X$  such that  $Fa, Fb$  are linearly independent. In particular, the vectors  $a, b$  are linearly independent. Since  $F$  is algebraic, there exists a non-negative integer  $k$  such that  $F^{k+1}a$  and  $F^{k+1}b$  belong to  $\text{Span}\{F^i a, F^i b : 0 \leq i \leq k\}$ . Let  $Z$  be a closed subspace such that  $X = \text{Span}\{F^i a, F^i b : 0 \leq i \leq k\} \oplus Z$ . Relatively to this decomposition, the operator  $F$  can be expressed as follows

$$F = \begin{pmatrix} F_1 & F_2 \\ 0 & F_3 \end{pmatrix}.$$

Consider also the operator  $T \in \mathcal{B}(X)$  represented by the matrix

$$T = \begin{pmatrix} S - 2F_1 & 0 \\ 0 & I - 2F_3 \end{pmatrix},$$

where  $S$  is an invertible operator such that  $Sa = Fa$  and  $Sb = 2Fb$ . Clearly, it follows from the previous Lemma that  $T, T + F$  and  $T + 2F$  are algebraic operators. But, since  $Tb = (T + F)a = 0$ , we get that  $T$  and  $T + F$  belong to  $\mathcal{A}_s(X)$ . On the other hand, we have

$$T + 2F = \begin{pmatrix} S & 2F_2 \\ 0 & I \end{pmatrix},$$

and hence  $T + 2F$  is invertible by Lemma 2.5. Thus,  $T + 2F \notin \mathcal{A}_s(X)$ . This completes the proof of (ii).

Since the second assertion implies the first one, it remains to prove the first assertion in the case when  $F$  is rank one. Put  $F = z \otimes f$  where  $z \in X$  and  $f \in X'$  are non-zero. Let  $y$  be such that  $f(y) = 1$ . Perturbing  $y$  by a suitable vector of  $\ker(f)$ , we can suppose that  $y$  and  $z$  are linearly independent. Furthermore, since  $X = \text{Span}\{y, z\} + \ker(f)$ , there exists a closed subspace  $Y \subseteq \ker(f)$  such that  $X = \text{Span}\{y, z\} \oplus Y$ . Consider the operator  $T \in \mathcal{B}(X)$  given by

$$T|_Y = I, \quad Ty = 0 \quad \text{and} \quad Tz = y - 2Fz.$$

Clearly,  $T \in \mathcal{A}_s(X)$ . However, we have

$$\begin{cases} (T + 2F)|_Y = I, \\ (T + 2F)y = 2z \quad \text{and} \quad (T + 2F)z = y. \end{cases}$$

So that  $T + 2F$  is invertible, and hence  $T + 2F \notin \mathcal{A}_s(X)$ . This completes the proof. ■



In the following theorem we characterize the rank one operators in terms of algebraic invertible operators.

**THEOREM 2.6.** *Let  $F \in \mathcal{B}(X)$  be non-zero. The following assertions are equivalent:*

- (i)  $F$  is a rank one operator;
- (ii) for every  $T \in \mathcal{A}_1(X)$  we have  $T + F \in \mathcal{A}_1(X)$  or  $T - F \in \mathcal{A}_1(X)$ .

*Proof.* (i)  $\Rightarrow$  (ii) Suppose that  $F = z \otimes f$  is a rank one operator and let  $T \in \mathcal{A}_1(X)$ . Assume, on the contrary, that  $T + F$  and  $T - F$  do not belong to  $\mathcal{A}_1(X)$ . Since  $T + F$  and  $T - F$  are algebraic, then  $T + F$  and  $T - F$  are not invertible. Hence, using (2.1),  $f(T^{-1}z) = -1$  and  $f(T^{-1}z) = 1$ , a contradiction.

(ii)  $\Rightarrow$  (i) Assume that  $\dim \text{ran}(F) \geq 2$ . Consider vectors  $x_0, y_0$  such that  $Fx_0, Fy_0$  are linearly independent and put  $X = \text{Span}\{x_0, y_0, Fx_0, Fy_0\} \oplus Y$  where  $Y$  is a closed subspace. With respect to this decomposition, consider the operator  $T \in \mathcal{B}(X)$  given by  $T = S \oplus I$  where  $S$  is an invertible operator such that  $Sx_0 = -Fx_0$  and  $Sy_0 = Fy_0$ . It follows that  $T$  is algebraic invertible, but  $T + F$  and  $T - F$  are not injective. This completes the proof. ■

### 3. PROOF OF MAIN RESULT

Combining Theorem 2.4 with Proposition 2.2, we obtain:

**LEMMA 3.1.** *Let  $\Phi : \mathcal{B}(X) \rightarrow \mathcal{B}(X)$  be an additive surjective map preserving  $\mathcal{A}_s(X)$  in both directions. Then*

- (i)  $\Phi$  is injective;
- (ii)  $\Phi$  preserves the set of rank one operators in both directions.

*Proof.* (i) Assume on the contrary that there exists  $F \neq 0$  such that  $\Phi(F) = 0$ . Then  $F \in \mathcal{A}_s(X)$  and, by Theorem 2.4, there exists  $T \in \mathcal{A}_s(X)$  satisfying  $T + 2F \notin \mathcal{A}_s(X)$ . But,  $\Phi(T + 2F) = \Phi(T) \in \mathcal{A}_s(X)$ , the desired contradiction.

(ii) Let  $F \in \mathcal{A}_s(X)$  with  $\dim \text{ran}(F) \geq 2$ . Then it follows again by Theorem 2.4 that there exists  $T \in \mathcal{A}_s(X)$  such that  $T + F \in \mathcal{A}_s(X)$  and  $T + 2F \notin \mathcal{A}_s(X)$ . Thus,  $\Phi(T) \in \mathcal{A}_s(X)$ ,  $\Phi(T) + \Phi(F) \in \mathcal{A}_s(X)$  and  $\Phi(T) + 2\Phi(F) \notin \mathcal{A}_s(X)$ . Therefore, we obtain by Proposition 2.2 that  $\dim \text{ran}(\Phi(F)) \geq 2$ . On the

other hand, since  $\Phi$  is bijective and  $\Phi^{-1}$  satisfies the same properties as  $\Phi$ , we obtain that  $\Phi$  preserves the set of rank one operators in both directions. ■

As a consequence of Theorem 2.6, we have the following lemma.

LEMMA 3.2. *Let  $\Phi : \mathcal{B}(X) \rightarrow \mathcal{B}(X)$  be an additive surjective map preserving  $\mathcal{A}_1(X)$  in both directions. Then*

- (i)  $\Phi$  is injective;
- (ii)  $\Phi$  preserves the set of rank one operators in both directions.

*Proof.* (i) Assume on the contrary that there exists  $F \neq 0$  such that  $\Phi(F) = 0$ . Then,  $F \notin \mathcal{A}_1(X)$ . It follows, in particular, that  $F$  is not of the form  $cI$  with  $c \neq 0$ , and so there exists  $y \in X$  such that the vectors  $y, Fy$  are linearly independent. Write  $X = \text{Span}\{y, Fy\} \oplus Z$  where  $Z$  is a closed subspace. Let  $T \in \mathcal{A}_1(X)$  defined by  $T = A \oplus I$  where  $A$  is an invertible operator such that  $Ay = -Fy$ . It follows that  $T + F$  is not injective and so  $T + F \notin \mathcal{A}_1(X)$ . This yields to a contradiction with  $\Phi(T + F) = \Phi(T) \in \mathcal{A}_1(X)$ .

(ii) Let  $F \in \mathcal{B}(X)$  be a rank one operator and let  $T \in \mathcal{A}_1(X)$ . Then  $T = \Phi(S)$  where  $S \in \mathcal{A}_1(X)$ . It follows by Theorem 2.6 that  $S + F \in \mathcal{A}_1(X)$  or  $S - F \in \mathcal{A}_1(X)$ , and hence  $T + \Phi(F) \in \mathcal{A}_1(X)$  or  $T - \Phi(F) \in \mathcal{A}_1(X)$ . Therefore, again by Theorem 2.6, we obtain that  $\Phi(F)$  is a rank one operator. Since  $\Phi$  is bijective and  $\Phi^{-1}$  satisfies the same properties as  $\Phi$ , we obtain that  $\Phi$  preserves the set of rank one operators in both directions. ■

LEMMA 3.3. *Let  $T, S \in \mathcal{A}_1(X)$  be such that*

$$T + F \in \mathcal{A}_1(X) \quad \Leftrightarrow \quad S + F \in \mathcal{A}_1(X),$$

*for every rank one operator  $F \in \mathcal{B}(X)$ . Then  $T = S$ .*

*Proof.* Note that since  $T$  and  $S$  are invertible, it suffices to show that  $T^{-1} = S^{-1}$ . Suppose, on the contrary, that there exists a vector  $x \in X$  such that  $T^{-1}x \neq S^{-1}x$ . Consider a linear form  $f \in X'$  such that  $f(T^{-1}x) = -1$  and  $f(S^{-1}x) \neq -1$ . Then we get by (2.1) that  $T + x \otimes f \notin \mathcal{A}_1(X)$  and  $S + x \otimes f \in \mathcal{A}_1(X)$ . This contradiction completes the proof. ■

Let  $\tau$  be a field automorphism of  $\mathbb{K}$ . An additive map  $A : X \rightarrow Y$  defined between two Banach spaces will be called  $\tau$ -semi linear if  $A(\lambda x) = \tau(\lambda)Ax$  holds for all  $x \in X$  and  $\lambda \in \mathbb{K}$ . If  $\tau$  is the complex conjugation, we will say

simply that  $A$  is *conjugate linear*. Notice that if  $A$  is non-zero and bounded, then  $\tau$  is continuous, and consequently,  $\tau$  is either the identity or the complex conjugation, see [10, Theorem 14.4.2 and Lemma 14.5.1]. Moreover, in this case, the adjoint operator  $A' : Y' \rightarrow X'$  defined by

$$A'(g) = \tau^{-1} \circ g \circ A \quad \text{for all } g \in Y',$$

is again  $\tau$ -semi linear.

LEMMA 3.4. *Let  $\Phi : \mathcal{B}(X) \rightarrow \mathcal{B}(X)$  be an additive surjective unital map preserving  $\mathcal{A}_1(X)$  in both directions. Then either*

- (i) *there exists an invertible bounded linear, or conjugate linear, operator  $A : X \rightarrow X$  such that  $\Phi(F) = AFA^{-1}$  for all finite rank operator  $F \in \mathcal{B}(X)$ , or*
- (ii) *there exists an invertible bounded linear, or conjugate linear, operator  $B : X' \rightarrow X$  such that  $\Phi(F) = BF'B^{-1}$  for all finite rank operator  $F \in \mathcal{B}(X)$ . In this case,  $X$  is reflexive.*

*Proof.* Lemma 3.2 ensures that  $\Phi$  is bijective and preserves the set of rank one operators in both directions. Then, by [13, Theorem 3.3], there exist a ring automorphism  $\tau : \mathbb{K} \rightarrow \mathbb{K}$  and either two bijective  $\tau$ -semi linear mappings  $A : X \rightarrow X$  and  $C : X' \rightarrow X'$  such that

$$\Phi(x \otimes f) = Ax \otimes Cf \quad \text{for all } x \in X \text{ and } f \in X', \tag{3.1}$$

or two bijective  $\tau$ -semi linear mappings  $B : X' \rightarrow X$  and  $D : X \rightarrow X'$  such that

$$\Phi(x \otimes f) = Bf \otimes Dx \quad \text{for all } x \in X \text{ and } f \in X'. \tag{3.2}$$

Suppose that  $\Phi$  satisfies (3.1), and let us show that

$$C(f)(Ax) = \tau(f(x)) \quad \text{for all } x \in X \text{ and } f \in X'. \tag{3.3}$$

Clearly, it suffices to establish that for all  $x \in X$  and  $f \in X'$ ,

$$f(x) = -1 \quad \text{if and only if} \quad C(f)(Ax) = -1.$$

Let  $x \in X$  and  $f \in X'$ . Then

$$\begin{aligned} f(x) = -1 &\Leftrightarrow I + x \otimes f \notin \mathcal{A}_1(X) \\ &\Leftrightarrow I + Ax \otimes Cf \notin \mathcal{A}_1(X) \\ &\Leftrightarrow C(f)(Ax) = -1. \end{aligned}$$

Thus, equation (3.3) holds, and arguing as in [13], we get that  $\tau$ ,  $A$ ,  $C$  are continuous,  $\tau$  is the identity or the complex conjugation, and  $C = (A^{-1})'$ . Therefore,  $\tau^{-1} = \tau$  and, for every  $u \in X$ , we have

$$\Phi(x \otimes f)u = \tau(fA^{-1}u)Ax = A(f(A^{-1}u)x) = A(x \otimes f)A^{-1}u.$$

Thus,  $\Phi(x \otimes f) = A(x \otimes f)A^{-1}$  for all  $x \in X$  and  $f \in X'$ . Hence,  $\Phi(F) = AFA^{-1}$  for all finite rank operator  $F \in \mathcal{B}(X)$ .

Now, suppose that  $\Phi$  satisfies (3.2), and let us show that

$$D(x)(Bf) = \tau(f(x)) \text{ for all } x \in X \text{ and } f \in X'. \quad (3.4)$$

Let  $x \in X$  and  $f \in X'$ . Then

$$\begin{aligned} f(x) = -1 &\Leftrightarrow I + x \otimes f \notin \mathcal{A}_i(X) \\ &\Leftrightarrow I + Bf \otimes Dx \notin \mathcal{A}_i(X) \\ &\Leftrightarrow D(x)(Bf) = -1. \end{aligned}$$

Thus, equation (3.4) holds, and arguing as in [13], we get that  $\tau$ ,  $B$ ,  $D$  are continuous,  $\tau$  is the identity or the complex conjugation, and  $D = (B^{-1})'J$ , where  $J : X \rightarrow X''$  is the natural embedding. But, the operators  $D$  and  $(B^{-1})'$  and therefore also  $J$  are bijections which implies the reflexivity of  $X$ . Furthermore,  $\tau^{-1} = \tau$  and, for every  $u \in X$ , we have

$$\begin{aligned} \Phi(x \otimes f)u &= (Bf \otimes (B^{-1})'J(x))u = (B^{-1})'J(x)(u)Bf \\ &= \tau(J(x)(B^{-1}u))Bf = B(J(x)(B^{-1}u)f) \\ &= B(f \otimes J(x))B^{-1}u = B(x \otimes f)'B^{-1}u. \end{aligned}$$

Thus,  $\Phi(x \otimes f) = B(x \otimes f)'B^{-1}$  for all  $x \in X$  and  $f \in X'$ . Hence,  $\Phi(F) = BF'B^{-1}$  for all finite rank operator  $F \in \mathcal{B}(X)$ . This completes the proof. ■

With these results at hand, we are ready to prove our main theorem.

*Proof of Theorem A.* (i)  $\Rightarrow$  (ii) Suppose that  $\Phi$  preserves  $\mathcal{A}_s(X)$  in both directions. It follows from Lemma 3.1 that  $\Phi$  is bijective and preserves the set of rank one operators in both directions. Let  $T \in \mathcal{A}_i(X)$ , and consider  $z \in X$  and  $f \in X'$  such that  $f(T^{-1}z) = -1$ . Then  $T + z \otimes f$  is not invertible by (2.1), and so  $T + z \otimes f \in \mathcal{A}_s(X)$ . Thus,  $\Phi(T) + \Phi(z \otimes f) \in \mathcal{A}_s(X)$ , and consequently  $\Phi(T)$  is algebraic because  $\Phi(z \otimes f)$  is rank one. But, since  $T \notin \mathcal{A}_s(X)$ , we have  $\Phi(T) \notin \mathcal{A}_s(X)$ , and hence  $\Phi(T) \in \mathcal{A}_i(X)$ .

Since  $\Phi$  is bijective and  $\Phi^{-1}$  satisfies the same properties as  $\Phi$ , we obtain that  $\Phi$  preserves  $\mathcal{A}_i(X)$  in both directions.

(ii)  $\Rightarrow$  (iii) Suppose that  $\Phi$  preserves  $\mathcal{A}_i(X)$  in both directions. It follows that  $\Phi$  takes one of the two forms in Lemma 3.4.

Assume that  $\Phi(F) = AFA^{-1}$  for all finite rank operator  $F \in \mathcal{B}(X)$ . Let

$$\Psi(T) = A^{-1}\Phi(T)A \quad \text{for all } T \in \mathcal{B}(X).$$

Clearly,  $\Psi$  satisfies the same properties as  $\Phi$ . Furthermore,  $\Psi(I) = I$  and  $\Psi(F) = F$  for all rank one operator  $F \in \mathcal{B}(X)$ . Let  $T \in \mathcal{B}(X)$  be an algebraic operator. Choose an arbitrary rational number  $\lambda$  such that  $T - \lambda$  and  $\Psi(T) - \lambda$  are invertible. We have

$$T - \lambda + F \in \mathcal{A}_i(X) \Leftrightarrow \Psi(T - \lambda) + F \in \mathcal{A}_i(X),$$

for all rank one operator  $F \in \mathcal{B}(X)$ . Hence, we get by Lemma 3.3 that  $\Psi(T) = T$ . This shows that  $\Phi(T) = ATA^{-1}$  for all  $T \in \mathcal{A}_o(X)$ .

Now suppose that  $\Phi(F) = BF'B^{-1}$  for all finite rank operator  $F \in \mathcal{B}(X)$ . Then, Lemma 3.4 ensures that  $X$  is reflexive. By considering

$$\Gamma(T) = J^{-1}(B^{-1}\Phi(T)B)'J \quad \text{for all } T \in \mathcal{B}(X),$$

we get in similar way that  $\Gamma(T) = T$  for all  $T \in \mathcal{A}_o(X)$ . Thus,  $\Phi(T) = BT'B^{-1}$  for all  $T \in \mathcal{A}_o(X)$ , as desired.

(iii)  $\Rightarrow$  (i) Is obvious. ■

We close this paper by the following remark and question.

*Remark 3.5.* Let  $X$  and  $Y$  be infinite-dimensional complex Banach spaces. Theorem A can be without any change formulated for unital additive surjective mappings  $\Phi : \mathcal{B}(X) \rightarrow \mathcal{B}(Y)$  preserving non-invertible algebraic operators in both directions.

**QUESTION 3.6.** It would be interesting to know if an analogue result of Theorem A can be obtained without supposing  $\Phi$  unital.

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